微分積分I問題集の補足

p.7

1. (7) $x \to 2+0$ より $x > 2$. よって $|x-2| = x-2$. (8)$x \to 3-0$ より $x < 3$. よって $|x-3| = -(x-3)$.

3. (1) $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ を求める. (2) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ を求める.

7. (2) $V = \pi r^2 h$, $S = 2\pi r^2 + 2\pi rh$. h は一定 (定数) なので r の 2 次関数として微分.

   (3) では h の 1 次式として微分. S の $2\pi r^2$ は定数項なので微分すると 0.

8. (6) $x = -t$ よって $t = -x$ とおくとよい. 解答参照. このとき $x \to -\infty$ のとき $t \to \infty$

9. (1) $y = (2x^2-1)\sqrt[4]{2x^2-1} = (2x^2-1)^1(2x^2-1)^{\frac{1}{4}} = (2x^2-1)^{\frac{5}{4}}$

   (5) $y = (2x-1)\sqrt[3]{3x+1} = (2x-1)(3x+1)^{\frac{1}{3}}$.

   $y' = (2x-1)'(3x+1)^{\frac{1}{3}} + (2x-1)\{(3x+1)^{\frac{1}{3}}\}' = 2(3x+1)^{\frac{1}{3}} + (2x-1)\cdot\dfrac{1}{3}(3x+1)^{-\frac{2}{3}}(3x+1)'$

   $= 2(3x+1)^{\frac{1}{3}} + \dfrac{(2x-1)\cdot 3}{3(3x+1)^{\frac{2}{3}}} = \dfrac{6(3x+1)+2x-1}{(3x+1)^{\frac{2}{3}}} = \cdots$

   (6) $y = (x-1)^{\frac{1}{2}}(x+1)^{-\frac{1}{2}}$. $y' = \dfrac{1}{2}(x-1)^{-\frac{1}{2}}(x+1)^{-\frac{1}{2}} + (x-1)^{\frac{1}{2}}\cdot\left(-\dfrac{1}{2}\right)(x+1)^{-\frac{3}{2}}$

   $= \dfrac{1}{2\sqrt{x-1}\sqrt{x+1}} - \dfrac{\sqrt{x-1}}{2\sqrt{x+1}^3} = \dfrac{(x+1)-(x-1)}{2\sqrt{x-1}\sqrt{x+1}^3} = \dfrac{1}{\sqrt{(x-1)(x+1)^3}}$

10. (1) $y' = \left\{\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{\frac{1}{3}}\right\}' = \dfrac{1}{3}\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{-\frac{2}{3}}\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)'$. ここで

    $$\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)' = \frac{(1-\sqrt{x})'(1+\sqrt{x}) - (1-\sqrt{x})(1+\sqrt{x})'}{(1+\sqrt{x})^2} = \frac{-\dfrac{1}{2\sqrt{x}}(1+\sqrt{x}) - (1-\sqrt{x})\dfrac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2}$$

    $= \dfrac{-1-\sqrt{x}-1+\sqrt{x}}{2\sqrt{x}(1+\sqrt{x})^2} = \dfrac{-2}{2\sqrt{x}(1+\sqrt{x})^2} = -\dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}$. よって

    $y' = \dfrac{1}{3}\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)^{\frac{2}{3}}\left(-\dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}\right) = -\dfrac{1}{3\sqrt{x}(1-\sqrt{x})^{\frac{2}{3}}(1+\sqrt{x})^{\frac{4}{3}}}$.

    (2) $y' = \left\{\left(\dfrac{1-x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right\}' = \dfrac{1}{2}\left(\dfrac{1-x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}\right)^{-\frac{1}{2}}\left(\dfrac{1-x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}\right)'$. ここで

    $$\left(\frac{1-x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}\right)' = \frac{(1-x^{\frac{1}{3}})'(1+x^{\frac{1}{3}}) - (1-x^{\frac{1}{3}})(1+x^{\frac{1}{3}})'}{(1+x^{\frac{1}{3}})^2} = \frac{-\dfrac{1}{3}x^{-\frac{2}{3}}(1+x^{\frac{1}{3}}) - (1-x^{\frac{1}{3}})\dfrac{1}{3}x^{-\frac{2}{3}}}{(1+x^{\frac{1}{3}})^2}$$

    $= \dfrac{-1-x^{\frac{1}{3}}-1+x^{\frac{1}{3}}}{3x^{\frac{2}{3}}(1+x^{\frac{1}{3}})^2} = \dfrac{-2}{3x^{\frac{2}{3}}(1+x^{\frac{1}{3}})^2}$. 故に

    $y' = \dfrac{1}{2}\left(\dfrac{1+x^{\frac{1}{3}}}{1-x^{\frac{1}{3}}}\right)^{\frac{1}{2}}\dfrac{-2}{3x^{\frac{2}{3}}(1+x^{\frac{1}{3}})^2} = -\dfrac{1}{3x^{\frac{2}{3}}(1-x^{\frac{1}{3}})^{\frac{1}{2}}(1+x^{\frac{1}{3}})^{\frac{3}{2}}}$.

11. (1) $f'(0)$ が存在すれば $x=0$ において微分可能. $f'(0) = \lim_{h \to 0}\dfrac{f(0+h)-f(0)}{h}$. ここで

    $\lim_{h \to +0}\dfrac{f(0+h)-f(0)}{h} = \lim_{h \to +0}\dfrac{|h^3(h-1)|-0}{h} = \lim_{h \to +0}\dfrac{-h^3(h-1)}{h} = \lim_{h \to +0}\{-h^2(h-1)\} = 0$

    $\lim_{h \to -0}\dfrac{f(0+h)-f(0)}{h} = \lim_{h \to -0}\dfrac{|h^3(h-1)|-0}{h} = \lim_{h \to -0}\dfrac{h^3(h-1)}{h} = \lim_{h \to +0} h^2(h-1) = 0$

    故に $f'(0) = 0$ で $x = 0$ において微分可能.

(2) $\lim_{x\to 1} f(x) = f(1)$ ならば $x=1$ において連続. ここで

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0} |x^3(x-1)| = \lim_{x\to 1+0} x^3(x-1) = 0.$$

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} |x^3(x-1)| = \lim_{x\to 1-0} \{-x^3(x-1)\} = 0.$$

よって $\lim_{x\to 1} f(x) = 0 = f(1)$. $x=1$ において連続.

$f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ より

$$\lim_{h\to +0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to +0} \frac{|(1+h)^3 h| - 0}{h} = \lim_{h\to +0} \frac{(1+h)^3 h}{h} = \lim_{h\to +0} \{(1+h)^3 = 1$$

$$\lim_{h\to -0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to -0} \frac{|(1+h)^3 h| - 0}{h} = \lim_{h\to -0} \frac{-(1+h)^3 h}{h} = \lim_{h\to +0} \{-(1+h)^3\} = -1$$

よって $f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ は存在しない, $x=1$ において微分可能でない.

14. (3) $\displaystyle \lim_{h\to 0} \frac{1}{h} \left\{ \frac{f(a+h)}{a+h} - \frac{f(a-h)}{a-h} \right\} = \lim_{h\to 0} \frac{1}{h} \frac{(a-h)f(a+h) - (a+h)f(a-h)}{(a+h)(a-h)}$

$$= \lim_{h\to 0} \frac{af(a+h) - hf(a+h) - af(a-h) - hf(a-h)}{h(a+h)(a-h)}$$

$$= \lim_{h\to 0} \left\{ \frac{af(a+h) - af(a-h)}{h(a+h)(a-h)} + \frac{-hf(a+h) - hf(a-h)}{h(a+h)(a-h)} \right\}$$

$$= \lim_{h\to 0} \left\{ \frac{a}{(a+h)(a-h)} \cdot \frac{f(a+h) - f(a-h)}{h} - \frac{f(a+h) + f(a-h)}{(a+h)(a-h)} \right\}$$

$$= \frac{a}{a^2} \lim_{h\to 0} \left\{ \frac{f(a+h) - f(a-h)}{h} \right\} - \frac{f(a) + f(a)}{a^2}$$

$$= \frac{a}{a^2} \lim_{h\to 0} \left\{ \frac{f(a+h) - f(a) + f(a) - f(a-h)}{h} \right\} - \frac{2f(a)}{a^2}$$

$$= \frac{a}{a^2} \left\{ f'(a) + \lim_{h\to 0} \frac{f(a) - f(a-h)}{h} \right\} - \frac{2f(a)}{a^2}$$

$$= \frac{a}{a^2} \left\{ f'(a) + \lim_{h\to 0} \frac{-f(a) + f(a-h)}{-h} \right\} - \frac{2f(a)}{a^2} \quad (-h = k \text{ とおくと } k \to 0)$$

$$= \frac{a}{a^2} \left\{ f'(a) + \lim_{k\to 0} \frac{f(a+k) - f(a)}{k} \right\} - \frac{2f(a)}{a^2}$$

$$= \frac{a}{a^2} \{ f'(a) + f'(a) \} - \frac{2f(a)}{a^2} = \frac{2\{af'(a) - f(a)\}}{a^2}$$