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119. (1) $t=0$ のとき $x=0, y=1$. $t=1$ のとき $x=1, y=0$. よって曲線と x 軸, y 軸で囲まれる.

$$S=\int_0^1\left|y\frac{dx}{dt}\right|dt=\int_0^1|(t^2-2t+1)(t^2)'|dt=\int_0^1(t^2-2t+1)2tdt=\cdots$$

(2) $t=0$ のとき $x=0, y=1$. $t=\dfrac{\pi}{4}$ のとき $x=1, y=\dfrac{\sqrt{2}}{2}+1$. よって曲線と x 軸, y 軸, 直線 $x=1$ で囲まれる.

$$S=\int_0^{\frac{\pi}{4}}\left|y\frac{dx}{dt}\right|dt=\int_0^{\frac{\pi}{4}}|(\sin t+1)(\tan t)'|dt=\int_0^{\frac{\pi}{4}}\frac{\sin t+1}{\cos^2 t}dt=\int_0^{\frac{\pi}{4}}\frac{\sin t}{\cos^2 t}dt+\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2 t}dt.$$

1 番目の積分は $\cos t=u$ とおく.

(3) $t=0$ のとき $x=1, y=2$. $t=\dfrac{\pi}{2}$ のとき $x=0, y=0$. よって曲線と x 軸, 直線 $x=1$ で囲まれる.

$$S=\int_0^{\frac{\pi}{2}}\left|y\frac{dx}{dt}\right|dt=\int_0^{\frac{\pi}{2}}|(\cos 2t+1)(\cos t)'|dt=\int_0^{\frac{\pi}{2}}(\cos 2t+1)\sin tdt.$$

半角の公式より $\dfrac{1+\cos 2t}{2}=\cos^2 t$. $\therefore \cos 2t+1=2\cos^2 t$ と変形してから $\cos t=u$ とおく.

120. (1) $\dfrac{dx}{dt}=(3t^2)'=6t, \dfrac{dy}{dt}=(3t-t^3)'=3-3t^2$. よって $l=\displaystyle\int_0^{\sqrt{3}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$

$$=\int_0^{\sqrt{3}}\sqrt{(6t)^2+(3-3t^2)^2}dt=\int_0^{\sqrt{3}}\sqrt{9t^4+18t^2+9}dt=\int_0^{\sqrt{3}}3(t^2+1)dt=\cdots.$$

(2) $\dfrac{dx}{dt}=(\cos t+t\sin t)'=-\sin t+\sin t+t\cos t=t\cos t, \dfrac{dy}{dt}=(\sin t-t\cos t)'=\cos t-\cos t+t\sin t=t\sin t$.

よって $l=\displaystyle\int_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=\int_0^{\pi}\sqrt{(t\cos t)^2+(t\sin t)^2}dt=\int_0^{\pi}\sqrt{t^2(\cos^2 t+\sin^2 t)}dt=\int_0^{\pi}tdt.$

(3) $\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=(-2\sin t+2\sin 2t)^2+(2\cos t-2\cos 2t)^2$

$$=4\sin^2 t-8\sin t\sin 2t+4\sin^2 2t+4\cos^2 t-8\cos t\cos 2t+4\cos^2 2t$$

$$=4(\sin^2 t+\cos^2 t)-8(\sin t\sin 2t+\cos t\cos 2t)+4(\sin^2 2t+\cos^2 2t)=4-8\cos(2t-t)+4=8-8\cos t.$$

(第 2 項の変形には加法定理を用いた.) 半角の定理より $1-\cos t=2\sin^2\dfrac{t}{2}$ だから

$$l=\int_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=\int_0^{\pi}\sqrt{8\cdot 2\sin^2\frac{t}{2}}dt=\int_0^{\pi}4\sin\frac{t}{2}dt=\left[-8\cos\frac{t}{2}\right]_0^{\pi}$$

121. (1) $\dfrac{dx}{dt}=(t^{\frac{1}{2}})'=\dfrac{1}{2}t^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{t}}$. $\therefore V=\pi\displaystyle\int_0^1 y^2\left|\frac{dx}{dt}\right|dt=\pi\int_0^1(\sqrt{t}-t)^2\frac{1}{2\sqrt{t}}dt=\pi\int_0^1\frac{t-2t\sqrt{t}+t^2}{2\sqrt{t}}dt$

$$=\frac{1}{2}\pi\int_0^1(\sqrt{t}-2t+t\sqrt{t})dt$$

(2) $\therefore V=\pi\displaystyle\int_0^{\frac{\pi}{2}}y^2\left|\frac{dx}{dt}\right|dt=\pi\int_0^{\frac{\pi}{2}}\sin^2 2t\cos tdt=\pi\int_0^{\frac{\pi}{2}}(2\sin t\cos t)^2\cos tdt=4\pi\int_0^{\frac{\pi}{2}}\sin^2 t\cos^3 tdt$

(2 倍角の公式 $\sin 2t=2\sin t\cos t$ を用いた) $\cos^3 t$ のうち $\cos^2 t=1-\sin^2 t$ と置き換えて $\sin t=u$ とおく.

122. (1) $S=2\pi\displaystyle\int_0^{\frac{\pi}{2}}|y|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=2\pi\int_0^{\frac{\pi}{2}}3\sin t\sqrt{\{(\cos 2t)'\}^2+\{(3\sin t)'\}^2}dt$

$$=2\pi\int_0^{\frac{\pi}{2}}3\sin t\sqrt{(-2\sin 2t)^2+(3\cos t)^2}dt=6\pi\int_0^{\frac{\pi}{2}}\sin t\sqrt{16\sin^2 t\cos^2 t+9\cos^2 t}dt$$

$=6\pi\displaystyle\int_0^{\frac{\pi}{2}}\sin t\sqrt{16\sin^2 t+9}\cos tdt$ (2 倍角の公式 $\sin 2t=2\sin t\cos t$ を用いた)

$16\sin^2 t+9=u$ とおくと $32\sin t\cos tdt=du$.

t	0	$\to$	$\frac{\pi}{2}$
u	9	$\to$	25

$\therefore V = 6\pi \int_9^{25} \sqrt{u}\frac{1}{32}du$

(2) $\dfrac{dx}{dt} = (\cos t\sqrt{\cos 2t})' = -\sin t\sqrt{\cos 2t} + \cos t \cdot \dfrac{1}{2}(\cos 2t)^{-\frac{1}{2}}(\cos 2t)' = -\sin t\sqrt{\cos 2t} + \dfrac{-2\cos t\sin 2t}{2\sqrt{\cos 2t}}$

$= -\dfrac{\sin t\cos 2t + \cos t\sin 2t}{\sqrt{\cos 2t}} = -\dfrac{\sin(t+2t)}{\sqrt{\cos 2t}} = -\dfrac{\sin 3t}{\sqrt{\cos 2t}}$ (加法定理を用いた)

$\dfrac{dy}{dt} = (\sin t\sqrt{\cos 2t})' = \cos t\sqrt{\cos 2t} + \sin t \cdot \dfrac{1}{2}(\cos 2t)^{-\frac{1}{2}}(\cos 2t)' = \cos t\sqrt{\cos 2t} + \dfrac{-2\sin t\sin 2t}{2\sqrt{\cos 2t}}$

$= \dfrac{\cos t\cos 2t - \sin t\sin 2t}{\sqrt{\cos 2t}} = \dfrac{\cos(t+2t)}{\sqrt{\cos 2t}} = \dfrac{\cos 3t}{\sqrt{\cos 2t}}$ (加法定理を用いた)

$\therefore V = 2\pi\int_0^{\frac{\pi}{4}} \sin t\sqrt{\cos 2t}\sqrt{\left(-\dfrac{\sin 3t}{\sqrt{\cos 2t}}\right)^2 + \left(\dfrac{\cos 3t}{\sqrt{\cos 2t}}\right)^2}dt = 2\pi\int_0^{\frac{\pi}{4}} \sin t\sqrt{\cos 2t}\sqrt{\dfrac{\sin^2 3t + \cos^2 3t}{\cos 2t}}dt.$

$= 2\pi\int_0^{\frac{\pi}{4}} \sin t\sqrt{\cos 2t}\dfrac{1}{\sqrt{\cos 2t}}dt == 2\pi\int_0^{\frac{\pi}{4}} \sin t dt.$

123. (1)$S = \dfrac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta = \dfrac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta = \dfrac{1}{2}\int_0^{\frac{\pi}{2}} 4\sin^2\theta\cos^2\theta d\theta = 2\int_0^{\frac{\pi}{2}} \sin^2\theta(1-\sin^2\theta)d\theta$

$= 2\left(\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta - \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta\right) = 2\left(\dfrac{1}{2}\dfrac{\pi}{2} - \dfrac{3}{4}\dfrac{1}{2}\dfrac{\pi}{2}\right) = \dfrac{\pi}{8}$

(2) $S = \dfrac{1}{2}\int_0^{2\pi} (\cos\theta + 2)^2 d\theta = \dfrac{1}{2}\int_0^{2\pi} (\cos^2\theta + 4\cos\theta + 4)d\theta = \dfrac{1}{2}\int_0^{2\pi}\left(\dfrac{1+\cos 2\theta}{2} + 4\cos\theta + 4\right)d\theta.$

124. (1)$l = \int_0^{\pi}\sqrt{r^2 + (r')^2}d\theta = \int_0^{\pi}\sqrt{e^{2\theta} + e^{2\theta}}d\theta = \int_0^{\pi}\sqrt{2}e^{\theta}d\theta.$

(2) $l = \int_0^{\frac{\pi}{2}}\sqrt{(\cos^2\theta)^2 + (-2\cos\theta\sin\theta)^2}d\theta = \int_0^{\frac{\pi}{2}}\sqrt{\cos^4\theta + 4\cos^2\theta\sin^2\theta}d\theta = \int_0^{\frac{\pi}{2}}\sqrt{\cos^2\theta + 4\sin^2\theta}\cos\theta d\theta.$

$= \int_0^{\frac{\pi}{2}}\sqrt{1 + 3\sin^2\theta}\cos\theta d\theta.$ $\sin\theta = t$ とおくと

$l = \int_0^1 \sqrt{1+3t^2}dt = \int_0^1 \sqrt{3}\sqrt{\dfrac{1}{3} + t^2}dt = \dfrac{\sqrt{3}}{2}\left[t\sqrt{t^2 + \dfrac{1}{3}} + \dfrac{1}{3}\log\left|t + \sqrt{t^2 + \dfrac{1}{3}}\right|\right]_0^1$

$= \dfrac{\sqrt{3}}{2}\left\{\sqrt{\dfrac{4}{3}} + \dfrac{1}{3}\log\left(1 + \sqrt{\dfrac{4}{3}}\right) - \dfrac{1}{3}\log\sqrt{\dfrac{1}{3}}\right\} = \dfrac{\sqrt{3}}{2}\left\{\dfrac{2}{\sqrt{3}} + \dfrac{1}{3}\log\left(1 + \dfrac{2}{\sqrt{3}}\right) - \dfrac{1}{3}\log\dfrac{1}{\sqrt{3}}\right\}$

$= 1 + \dfrac{\sqrt{3}}{6}\log\dfrac{1 + \frac{2}{\sqrt{3}}}{\frac{1}{\sqrt{3}}} = 1 + \dfrac{\sqrt{3}}{6}\log(\sqrt{3} + 2)$

125. (1) $\int_0^2 |e^{-t}|dt = \int_0^2 e^{-t}dt.$ (2) $\int_1^2 |2\sin\pi t|dt = \int_1^2 (-2\sin\pi t)dt.$

126. (1) 与式 $= \lim_{\varepsilon\to+0}\int_{\varepsilon}^2 \dfrac{1}{x^3}dx = \lim_{\varepsilon\to+0}\int_{\varepsilon}^2 x^{-3}dx = \lim_{\varepsilon\to+0}\left[-\dfrac{1}{2}x^{-2}\right]_{\varepsilon}^2 = \lim_{\varepsilon\to+0}\dfrac{1}{2}\left(-\dfrac{1}{4} + \dfrac{1}{\varepsilon^2}\right) = \infty.$ よって存在しない.

(2) 与式 $= \lim_{\varepsilon\to+0,\varepsilon'\to+0}\int_{\varepsilon}^{1-\varepsilon'}\dfrac{dx}{\sqrt{x(1-x)}}.$ $\int\dfrac{dx}{\sqrt{x(1-x)}} = \int\dfrac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}}.$ $x - \dfrac{1}{2} = t$ とおくと

$\int\dfrac{dx}{\sqrt{x(1-x)}} = \int\dfrac{dt}{\sqrt{\frac{1}{4} - t^2}} = \mathrm{Sin}^{-1}\dfrac{t}{\frac{1}{2}} = \mathrm{Sin}^{-1}2t = \mathrm{Sin}^{-1}(2x-1).$ よって

与式 $= \lim_{\varepsilon\to+0,\varepsilon'\to+0}[\mathrm{Sin}^{-1}(2x-1)]_{\varepsilon}^{1-\varepsilon'} = \lim_{\varepsilon\to+0,\varepsilon'\to+0}\{\mathrm{Sin}^{-1}(1-2\varepsilon') - \mathrm{Sin}^{-1}(2\varepsilon - 1)\}$

$= \mathrm{Sin}^{-1}1 - \mathrm{Sin}^{-1}(-1) = \dfrac{\pi}{2} - \left(\dfrac{\pi}{2}\right) = \pi.$

(3) 与式 $= \lim_{b\to\infty}\int_0^b xe^{-x}dx = \lim_{b\to\infty}\left\{[-xe^{-x}]_0^b - \int_0^b(-e^{-x})dx\right\} = \lim_{b\to\infty}(-be^{-b} - [e^{-x}]_0^b) = \lim_{b\to\infty}(-be^{-b} - e^{-b} + 1).$

$\lim_{b\to\infty}e^{-b} = \lim_{b\to\infty}\dfrac{1}{e^b} = 0.$ ロピタルの定理より $\lim_{b\to\infty}be^{-b} = \lim_{b\to\infty}\dfrac{b}{e^b} = \lim_{b\to\infty}\dfrac{1}{e^b} = 0.$ よって 与式 $= 1.$

(4) 与式 $= \lim_{b\to\infty}\int_1^b \dfrac{1}{x(x^2+1)}dx = \lim_{b\to\infty}\int_1^b\left(\dfrac{1}{x} - \dfrac{x}{x^2+1}\right)dx = \lim_{b\to\infty}\left[\log|x| - \dfrac{1}{2}\log|x^2+1|\right]_1^b$

$= \lim_{b\to\infty}\left(\log b - \dfrac{1}{2}\log(b^2+1) - \log 1 + \dfrac{1}{2}\log 2\right) = \lim_{b\to\infty}(\log b - \log\sqrt{b^2+1}) + \log\sqrt{2} = \lim_{b\to\infty}\log\dfrac{b}{\sqrt{b^2+1}} + \log\sqrt{2}$

$$= \lim_{b\to\infty} \log \frac{1}{\sqrt{1+\frac{1}{b^2+1}}} + \log\sqrt{2} = \log 1 + \log\sqrt{2} = \log\sqrt{2}.$$

(5) 与式 $= \displaystyle\lim_{\varepsilon\to+0, b\to\infty} \int_{\varepsilon}^{b} \frac{dx}{\sqrt[3]{x^2}} = \lim_{\varepsilon\to+0, b\to\infty} \int_{\varepsilon}^{b} x^{-\frac{2}{3}} dx = \lim_{\varepsilon\to+0, b\to\infty} \left[3x^{\frac{1}{3}}\right]_{\varepsilon}^{b} = \lim_{\varepsilon\to+0, b\to\infty} \left(3b^{\frac{1}{3}} - 3\varepsilon^{\frac{1}{3}}\right) = \infty.$

よって存在しない.

(6) 与式 $= \displaystyle\lim_{\varepsilon\to+0, \varepsilon'\to+0} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x^4} + \int_{\varepsilon'}^{1} \frac{dx}{x^4}\right) = \lim_{\varepsilon\to+0} \int_{-1}^{-\varepsilon} x^{-4} dx + \lim_{\varepsilon'\to+0} \int_{\varepsilon'}^{1} x^{-4} dx$

$$= \lim_{\varepsilon\to+0} \left[-\frac{1}{3}x^{-3}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon'\to+0} \left[-\frac{1}{3}x^{-3}\right]_{\varepsilon'}^{1} = \lim_{\varepsilon\to+0} \left(-\frac{1}{3}(-\varepsilon)^{-3} + \frac{1}{3}(-1)^{-3}\right) + \lim_{\varepsilon'\to+0} \left(-\frac{1}{3} + \frac{1}{3}(\varepsilon')^{-3}\right).$$

$= \displaystyle\lim_{\varepsilon\to+0} \frac{1}{3\varepsilon^3} - \frac{1}{3} - \frac{1}{3} + \lim_{\varepsilon'\to+0} \frac{1}{3\varepsilon'^3} = \infty.$ よって存在しない.