第 3 章§ 2 変数の変換と重積分

p.93 練習問題 2-A

1. $x = X\cos\frac{\pi}{4} - Y\sin\frac{\pi}{4} = \frac{X-Y}{\sqrt{2}},\ y = X\sin\frac{\pi}{4} + Y\cos\frac{\pi}{4} = \frac{X+Y}{\sqrt{2}}$ より

$x+y = \frac{X-Y}{\sqrt{2}} + \frac{X+Y}{\sqrt{2}} = \frac{2X}{\sqrt{2}} = \sqrt{2}X,\ x-y = \frac{X-Y}{\sqrt{2}} - \frac{X+Y}{\sqrt{2}} = -\frac{2Y}{\sqrt{2}} = -\sqrt{2}Y$

$\therefore D : -\sqrt{2} \leqq \sqrt{2}X \leqq \sqrt{2},\ -\sqrt{2} \leqq -\sqrt{2}Y \leqq \sqrt{2} \Rightarrow -1 \leqq X \leqq 1, -1 \leqq Y \leqq 1$

与式 $= \iint_D (\sqrt{2}X)^2 dXdY = \int_{-1}^{1}\left\{\int_{-1}^{1} 2X^2 dY\right\}dX = \int_{-1}^{1}\left[2X^2Y\right]_{-1}^{1}dx = 2\int_0^1 4X^2 dx = 2\left[\frac{4X^3}{3}\right]_0^1 = \underline{\frac{8}{3}}$

2. $D : 0 \leqq r \leqq a,\ 0 \leqq \theta \leqq \frac{\pi}{2}$ また $x^2+y^2 = r^2$ より $\sqrt{x^2+y^2} = r$ よって

与式 $= \iint_D r \cdot r dr d\theta = \int_0^{\frac{\pi}{2}}\left\{\int_0^a r^2 dr\right\}d\theta = \int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^a d\theta = \int_0^{\frac{\pi}{2}}\frac{a^3}{3}d\theta = \frac{a^3}{3}[\theta]_0^{\frac{\pi}{2}} = \underline{\frac{\pi a^3}{6}}$

3. $z_x = x,\ z_y = y$ より $S = \iint_D \sqrt{(z_x)^2+(z_y)^2+1}\,dxdy = \iint_D \sqrt{x^2+y^2+1}\,dxdy$ 極座標に変換すると

$D : 0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq 2\pi$ よって $S = \iint_D \sqrt{r^2+1}\,r dr d\theta = \int_0^{2\pi}\left\{\int_0^1 \sqrt{r^2+1}\,r dr\right\}d\theta$

ここで $r^2+1 = t$ とおくと $2rdr = dt,\ rdr = \frac{1}{2}dt$ よって $\int\sqrt{r^2+1}dr = \int\sqrt{t}\frac{1}{2}dt = \frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}} = \frac{1}{3}(r^2+1)^{\frac{3}{2}}$

$S = \int_0^{2\pi}\left[\frac{1}{3}(r^2+1)^{\frac{3}{2}}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{3}\left(2^{\frac{3}{2}}-1\right)d\theta = \frac{2\sqrt{2}-1}{3}[\theta]_0^{2\pi} = \underline{\frac{2\pi(2\sqrt{2}-1)}{3}}$

4. (1) $D : -2 \leqq x+y \leqq 2,\ -1 \leqq 2x-y \leqq 1,\ x+y = -2 \Rightarrow y = -x-2,\ x+y = 2 \Rightarrow y = -x+2,\ 2x-y =$

$-1 \Rightarrow y = 2x+1,\ 2x-y = 1 \Rightarrow y = 2x-1,$ 図は解答参照

(2) $x+y = u \cdots$①$,\ 2x-y = v \cdots$② より ①$+$②$\Rightarrow 3x = u+v \therefore x = \frac{u+v}{3},$

①$\times 2 -$②$\Rightarrow 3y = 2u-v \therefore y = \frac{2u-v}{3}$ だから $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$

$D : |u| \leqq 2,\ |v| \leqq 1 \Rightarrow -2 \leqq u \leqq 2,\ -1 \leqq v \leqq 1$ よって 与式 $= \iint_D u^2v^4|J|dudv$

$= \frac{1}{3}\int_{-2}^{2}\left\{\int_{-1}^{1} u^2v^4 dv\right\}du = \frac{4}{3}\int_0^2\left\{\int_0^1 u^2v^4 dv\right\}du = \frac{4}{3}\int_0^2 u^2\left[\frac{v^5}{5}\right]_0^1 du = \frac{4}{15}\left[\frac{u^3}{3}\right]_0^2 = \underline{\frac{32}{45}}.$

5. $\sqrt{x}+\sqrt{y} = 1$ より $\sqrt{y} = 1-\sqrt{x}$ よって $y = (1-\sqrt{x})^2,\ D : 0 \leqq x \leqq 1,\ 0 \leqq y \leqq (1-\sqrt{x})^2$

$\iint_D dxdy = \int_0^1\left\{\int_0^{(1-\sqrt{x})^2} dy\right\}dx = \int_0^1 [y]_0^{(1-\sqrt{x})^2}dx = \int_0^1\left(1-\sqrt{x}\right)^2 dx = \int_0^1\left(1-2\sqrt{x}+x\right)dx$

$= \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$

$\iint_D xdxdy = \int_0^1\left\{\int_0^{(1-\sqrt{x})^2} xdy\right\}dx = \int_0^1 x[y]_0^{(1-\sqrt{x})^2}dx = \int_0^1 x\left(1-\sqrt{x}\right)^2 dx = \int_0^1\left(x - 2x\sqrt{x} + x^2\right)dx$

$= \left[\frac{x^2}{2} - \frac{4}{5}x^{\frac{5}{2}} + \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{4}{5} + \frac{1}{3} = \frac{1}{30}$

$\iint_D ydxdy = \int_0^1\left\{\int_0^{(1-\sqrt{x})^2} ydy\right\}dx = \int_0^1\left[\frac{y^2}{2}\right]_0^{(1-\sqrt{x})^2} dx = \int_0^1\frac{(1-\sqrt{x})^4}{2}dx$ ここで $1-\sqrt{x} = t$ とおくと

$\sqrt{x} = 1-t,\ x = 1-2t+t^2$ よって $dx = (-2+2t)dt,$

x	0	1
t	1	0

だから

$$\iint_D y\,dxdy = \int_1^0 \frac{t^4}{2}(-2+2t)dt = \int_0^1 (t^4 - t^5)dt = \left[\frac{t^5}{5} - \frac{t^6}{6}\right]_0^1 = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$$

よって $\bar{x} = \bar{y} = \frac{\frac{1}{30}}{\frac{1}{6}} = \frac{1}{5}$. よって重心の座標は $\underline{\left(\frac{1}{5}, \frac{1}{5}\right)}$