第4章 微分法

§1 数列

(1) $\sum_{k=0}^{n}(3k+1)=\sum_{k=1}^{n}3k+\sum_{k=0}^{n}1=\frac{3n(n+1)}{2}+n+1=\frac{(n+1)(3n+2)}{2}$

(1) 公比 r 初項 a $\frac{a(1-r^8)}{1-r}=17\frac{a(1-r^4)}{1-r}$ $(1-r^8)=17(1-r^4)$

$1+r^4=17$ $r^4=16$ $r=\pm 2$, $\pm 2i$

(2) 初項 8 公比 r $8r^3=-1$ $r^3=\frac{-1}{8}$ $r=-\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2}$ $\omega^3=1$ $\omega\neq 1$

(3) 初項 16 第5項 -1 $16r^4=-1$ $r^4=-\frac{1}{16}$

∴ 実数には存在しない

3 $S_n=x+2x^2+3x^3+\quad +(n-1)x^{n-1}+nx^n$

$xS_n=x^2+2x^3+3x^4+\quad +(n-1)x^n+nx^{n+1}$

$(1-x)S_n=x+x^2+x^3+\quad +x^n-nx^{n+1}$

$=\frac{x(1-x^n)}{1-x}-nx^{n+1}=\frac{x\{1-(n+1)x^n+nx^{n+1}\}}{1-x}$

∴ $S_n=\frac{x\{1-(n+1)x^n+nx^{n+1}\}}{(1-x)^2}$ $\quad x\neq 1$

$S_n=\frac{n(n+1)}{2}$ $\quad x=1$

(1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14) (15) (16) (17) (18) (19)

4 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, 12, 13, 13

第 n 項 $a_n=n-\left[\frac{n-1}{3}\right]$ $\quad [x]$ xをこえない最大の整数

$\sum_{k=1}^{n}\left\{k-\left[\frac{k-1}{3}\right]\right\}=\frac{n(n+1)}{2}-\left[\frac{n-1}{3}\right]\cdot\frac{\{n-3+n-3\left[\frac{n-1}{3}\right]\}}{2}$

$n-a_n$; 0 0 0 1 1 1 2 2 2 3 3 3 4 4 4 5 5 5

- $\{a_n\}$, $\{b_n\}$ $\quad \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}=\begin{pmatrix} 2 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix}$

(1) $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 2 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}=\lambda_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 8 \\ 4 \end{pmatrix}=\lambda_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad$ ∴ $\lambda_1=4$

$\begin{pmatrix} 2 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}=\lambda_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\quad \begin{pmatrix} -6 \\ 12 \end{pmatrix}=\lambda_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\quad$ ∴ $\lambda_2=-6$

(2) $\begin{pmatrix} a_{ni} \\ b_{ni} \end{pmatrix} = \lambda_i^n \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ $\quad \begin{pmatrix} a_{n+1\,i} \\ b_{n+1\,i} \end{pmatrix} = \lambda_i^{n+1} \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \lambda_i^n \lambda_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$

$$= \lambda_i^n A \begin{pmatrix} x_i \\ y_i \end{pmatrix} = A \lambda_i^n \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

$$= A \begin{pmatrix} a_{ni} \\ b_{ni} \end{pmatrix}$$

$$\therefore \begin{pmatrix} a_{n+1\,i} \\ b_{n+1\,i} \end{pmatrix} = A \begin{pmatrix} a_{ni} \\ b_{ni} \end{pmatrix}$$

(3) $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} a_{n1} & a_{n2} \\ b_{n1} & b_{n2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $\quad \begin{pmatrix} a_{n1} \\ b_{n1} \end{pmatrix} = 4^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} a_{n2} \\ b_{n2} \end{pmatrix} = (-6)^n \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} a_{11} \\ b_{11} \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$ $\quad \begin{pmatrix} a_{12} \\ b_{12} \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \end{pmatrix}$ $\quad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} -10 \\ 40 \end{pmatrix}$

$\therefore \begin{pmatrix} -10 \\ 40 \end{pmatrix} = \begin{pmatrix} 8 & -6 \\ 4 & 12 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $\quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{120} \begin{pmatrix} 12 & 6 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} -10 \\ 40 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 2\cdot 4^n & (-6)^n \\ 4^n & -2(-6)^n \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 4^n + 3(-6)^n \\ 4^n - 6(-6)^n \end{pmatrix}$$

6 (1) $a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} {}_nC_k \left(\frac{1}{n}\right)^k = 1 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\leqq 1 + \sum_{k=1}^{n} \frac{1}{k!} = \sum_{k=0}^{n} \frac{1}{k!}$$

(2) $n>2$ に対して $n! > 2^{n-1}$ であること

[Ⅰ] $n=3$ のとき $3! = 6 > 4 = 2^2$

∴ 成り立つ

[Ⅱ] $n=k$ のとき $k! > 2^{k-1}$

$n=k+1$ のとき $(k+1)! > k\cdot k! > k\cdot 2^{k-1} > 2^k$ ($k>2$ より)

∴ $n=k+1$ のとき成り立つ

[Ⅰ].[Ⅱ] よりすべての $n>2$ に対して $n! > 2^{n-1}$

(3) (1).(2) より

$$a_n \leqq \sum_{k=0}^{n} \frac{1}{k!} \leqq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 1+2-\left(\frac{1}{2}\right)^{n-1} < 3$$

∴ $a_n < 3$

(4) (1) より $a_{n+1} - a_n = \sum_{k=1}^{n+1} \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) - \sum_{k=1}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$

> 0 $\quad \frac{1}{n+1} < \frac{1}{n}$

§2 級数

(1) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

(2) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)} - \frac{1}{n(n+2)}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$= \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) - \frac{1}{2}\lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$

(1) $f(x) = \log(1+x) \quad f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots \quad , |x|<1$

$\therefore -\frac{1}{x}\log(1-x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots + \frac{x^{n-1}}{n} + \cdots \quad 0<|x|<1$

$\therefore$ $0<|x|<1$ のとき

$\sum_{n=0}^{\infty} \frac{x^n}{n+1} = -\frac{1}{x}\log(1-x)$

$x=0$ のとき

$\sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1$

$x=-1$ のとき

$\sum_{n=0}^{\infty} \frac{x^n}{1+n} = \log 2$

他は発散

(2) $1 - \frac{2}{3} + \frac{4}{9} - \cdots = \sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n = \lim_{n\to\infty}\frac{1-\left(-\frac{2}{3}\right)^n}{1+\frac{2}{3}} = \frac{3}{5}$

(3) $\sum \frac{n}{n^3+1} \qquad \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ は収束 $\therefore$ $\sum \frac{n}{n^3+1}$ は収束

(4) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

$\{a_n\}$ $a_1 = 1$, $n \geq 2$ のとき $a_n = \frac{1}{2^{k+1}} \qquad 2^k < n \leq 2\cdot 2^k$

$2^k < 2^k+1,\ 2^k+2,\ \cdots, 2^k+2^k = 2^{k+1}$

$\therefore \sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2^0} + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}\right) + \frac{1}{2^4}$

$= 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ 発散

) $n \geq 3$ のとき $a_n = \frac{1}{2^{k+1}} \qquad n \leq 2^{k+1} \qquad \frac{1}{n} > \frac{1}{2^{k+1}}$

$\therefore a_n < b_n$

(3) $a_n \leqq b_n$ より $\sum a_n$ が発散 $\therefore \sum b_n$ 発散

2.4 (1) $n \geqq 2$ のとき $(1+\alpha)^n = 1 + n\alpha + \frac{n(n-1)}{2}\alpha^2 + \cdots + \alpha^n$

$$> \frac{n(n-1)}{2}\alpha^2$$

(2) (1) より

$$\frac{1}{(1+\alpha)^n} < \frac{2}{n(n-1)\alpha^2}$$

(a) $\lim_{n\to\infty} \frac{1}{(1+\alpha)^n} \leqq \lim_{n\to\infty} \frac{2}{n(n-1)\alpha^2} = 0.$ $\therefore \lim_{n\to\infty} \frac{1}{(1+\alpha)^n} = 0$

(b) $\lim_{n\to\infty} \frac{n}{(1+\alpha)^n} \leqq \lim_{n\to\infty} \frac{2}{(n-1)\alpha^2} = 0$ $\therefore \lim_{n\to\infty} \frac{n}{(1+\alpha)^n} = 0$

(3) (a) $\sum_{n=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$, $x \neq 1$ $x=1$ のとき $n+1$

(b) $S = \sum_{k=1}^{n} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}$

$xS = x + 2x^2 + 3x^3 + \cdots + (n-1)x^{n-1} + nx^n$

$(1-x)S = 1 + x + x^2 + \cdots + x^{n-1} - nx^n$

$S = \frac{1}{1-x}\left\{ \frac{1-x^n}{1-x} - nx^n \right\}$

$= \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$ $n \neq 1$

$S = \frac{n(n+1)}{2}$ $n = 1$

(4) (a) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $|x| < 1$ $|x| \geqq 1$ のとき発散

(b) $\sum_{n=1}^{\infty} n x^{n-1} = \frac{x}{(1-x)^2}$ $|x| < 1$ $|x| \geqq 1$ のとき発散

2.5 ダランベールの判定法 $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$

$r < 1$ のとき収束 $r > 1$ のとき発散

コーシーの判定法 $\lim_{n\to\infty} \sqrt[n]{a_n} = r$

$r < 1$ のとき収束 $r > 1$ のとき発散

2.6 $\displaystyle\sum \frac{n!(2n)!\,x^n}{(3n)!}$ の収束半径

$$\frac{a_n}{a_{n+1}} = \frac{n!(2n)!}{(3n)!}\frac{(3n+3)!}{(n+1)!(2n+2)!} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} \longrightarrow \frac{27}{4}$$

$\therefore\ R = \frac{27}{4}$

2.7 $\displaystyle\sum_{k=1}^{n} kx^k = x + 2x^2 + 3x^3 + \cdots + nx^n$

$$(1-x)\sum_{k=1}^{n} kx^k = x + x^2 + \cdots + x^n - nx^{n+1} = \frac{x(1-x^n)}{1-x} - nx^{n+1}$$

$$= \frac{x-(n+1)x^{n+1}+nx^{n+2}}{1-x}$$

$$\therefore\ \sum_{k=1}^{n} kx^k = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

$|x|<1$ のとき $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} kx^k = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$

$\therefore\ x=\frac{1}{2}$ のとき $\dfrac{\frac{1}{2}}{\frac{1}{4}} = 2$

2.8 $\displaystyle\sum_{k=0}^{n}(3x)^k = \frac{1-(3x)^{n+1}}{1-3x} \longrightarrow \frac{1}{1-3x}$ $\quad |3x|<1$ $(|x|<\frac{1}{3})$

$\therefore\ x=\frac{1}{4}$ のとき $\dfrac{1}{1-\frac{3}{4}} = 4$

§3 漸化式

(1) $a_1=4 \quad a_{n+1}=\dfrac{3a_n+2}{a_n+4}, \qquad a_{n+1}+2=\dfrac{5(a_n+2)}{a_n+4}$

$\therefore \dfrac{1}{a_{n+1}+2}=\dfrac{1}{5}\cdot\dfrac{a_n+4}{a_n+2}=\dfrac{1}{5}\left(1+\dfrac{2}{a_n+2}\right)=\dfrac{1}{5}+\dfrac{2}{5}\dfrac{1}{a_n+2}$

$\therefore b_n=\dfrac{1}{a_n+2}$ とおくと $b_1=\dfrac{1}{a_1+2}=\dfrac{1}{6}$

$b_{n+1}=\dfrac{1}{5}+\dfrac{2}{5}b_n \qquad b_{n+1}+\alpha=\dfrac{2}{5}(b_n+\alpha)$ とおくと

$-\dfrac{3}{5}\alpha=\dfrac{1}{5} \quad \alpha=-\dfrac{1}{3}$

$\therefore b_{n+1}-\dfrac{1}{3}=\dfrac{2}{5}\left(b_n-\dfrac{1}{3}\right)$ $\therefore \{b_n-\frac{1}{3}\}$ は公比 $\frac{2}{5}$ 初項 $-\frac{1}{6}$ の等比数列

$\therefore b_n-\dfrac{1}{3}=-\dfrac{1}{6}\left(\dfrac{2}{5}\right)^{n-1} \qquad \therefore b_n=\dfrac{1}{3}-\dfrac{2^{n-1}}{6\cdot 5^{n-1}}$

$a_n=\dfrac{1}{b_n}-2=\dfrac{1}{\frac{1}{3}-\frac{1}{6}\frac{2^{n-1}}{5^{n-1}}}-2$

$\therefore \lim\limits_{n\to\infty}a_n=3-2=1$

(2) $a_1=0 \quad a_{n+1}=\dfrac{1}{5}(a_n^2+1)$

(a) $\{a_n\}$ は単調増加で $a_n>0$ $(n\geqq 2)$ であること

$a_1=0 \quad a_2=\dfrac{1}{5}>0 \quad \therefore a_2-a_1>0$

$n=k$ のとき $a_{k+1}-a_k>0$ とすると

$a_{k+2}-a_{k+1}=\dfrac{1}{5}(a_{k+1}^2-a_k^2)>0$

$\therefore a_{k+2}>a_{k+1}$

(b) $a_n<1$ であること

$a_1<1 \qquad a_k<1$ とすると $a_{k+1}=\dfrac{1}{5}(a_k^2+1)<\dfrac{2}{5}<1$

(a),(b) より $\{a_n\}$ は単調増加で上方有界

$\therefore \lim\limits_{n\to\infty}a_n$ は存在する $\lim\limits_{n\to\infty}a_n=\alpha$ とおくと $\quad \alpha\leqq 1$

$n\to\infty$ のとき $a_{n+1}=\dfrac{1}{5}(a_n^2+1) \to \alpha=\dfrac{1}{5}(\alpha^2+1)$

$\therefore \alpha^2-5\alpha+1=0 \quad \therefore \alpha=\dfrac{5\pm\sqrt{21}}{2} \qquad \alpha\leqq 1$ より

$\alpha=\dfrac{5-\sqrt{21}}{2}$

2 $\sigma=\{(a_1\ a_2\ a_3 \cdots a_{10}) \mid a_{k+3}=3a_{k+2}+2a_{k+1}+a_k\}$

$a_4=3a_3+2a_2+a_1$

$a_5=3a_4+2a_3+a_2=3(3a_3+2a_2+a_1)+2a_3+a_2$

$=11a_3+7a_2+3a_1$

a_k : $k\geqq 4$ は $a_1\ a_2 . a_3$ の一次結合で表わされる

∴ σ は 3次元

$a_{n+1}=\sqrt{1+a_n}\quad a_1=1$

(1) $a_{n+1}-a_n=\sqrt{1+a_n}-\sqrt{1+a_{n-1}}=\dfrac{a_n-a_{n-1}}{\sqrt{1+a_n}+\sqrt{1+a_{n-1}}}$

(2) $a_2=\sqrt{1+a_1}=\sqrt{2}$

[I] ∴ $a_2-a_1>0$

[II] $n=k$ のとき $a_k-a_{k-1}>0$ とすると

$a_{k+1}-a_k=\dfrac{a_k-a_{k-1}}{\sqrt{1+a_k}+\sqrt{1+a_{k-1}}}>0$

∴ $a_{n+1}-a_n>0$ ∴ $\{a_n\}$ は単調増加

(3) $a_1=1<2$

$a_k<2$ とすると $a_{k+1}=\sqrt{1+a_k}<\sqrt{1+2}<2$ ∴ $a_{k+1}<2$

∴ $a_n<2$

(4) (2)(3) より

$\{a_n\}$ は単調増加で上方有界だから

$\lim_{n\to\infty} a_n$ は存在する $\lim_{n\to\infty} a_n=\alpha$ とおくと

$\alpha=\sqrt{1+\alpha}\qquad 0\leqq\alpha\leqq 2$

$\alpha^2-\alpha-1=0\qquad \alpha=\dfrac{1\pm\sqrt{5}}{2}\qquad ∴\ \alpha=\dfrac{1+\sqrt{5}}{2}$

3.4 $a_1 = 2 \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

(1) $\lim_{n\to\infty} a_n = \alpha$ とおくと $\quad \alpha = \frac{1}{2}\left(\alpha + \frac{2}{\alpha}\right) \quad \frac{1}{2}\alpha^2 - 1 = 0$

$\therefore \alpha = \pm\sqrt{2}. \quad \alpha > 0 \quad \therefore \alpha = \sqrt{2}$

(2) $\sqrt{2} < a_n$, $\{a_n\}$ が減少であること

$a_n > a_{n+1} > \sqrt{2}$

$n=1$ のとき $\quad a_1 = 2. \quad a_2 = \frac{3}{2} \quad \therefore a_1 > a_2 > \sqrt{2}$

$a_{k+1} = \frac{1}{2}\left(a_k + \frac{2}{a_k}\right) = \frac{1}{2}\left\{\left(\sqrt{a_k} - \frac{\sqrt{2}}{\sqrt{a_k}}\right)^2 + 2\sqrt{2}\right\} > \sqrt{2}$

$\therefore a_{k+1} > \sqrt{2}$

$a_{n+1} < a_n$ であること

$n=1$ のとき $\quad a_2 = \frac{3}{2} < 2 = a_1 \quad \therefore a_2 < a_1$

$n=k$ のとき $\quad a_{k+1} < a_k$ とすると

$n=k+1$ のとき $\quad a_{k+2} - a_{k+1} = \frac{1}{2}\left(a_{k+1} + \frac{2}{a_{k+1}}\right) - \frac{1}{2}\left(a_k + \frac{2}{a_k}\right)$

$= \frac{1}{2}\left(a_{k+1} - a_k\right) + \frac{1}{a_{k+1}} - \frac{1}{a_k}$

$= \left(a_{k+1} - a_k\right)\left\{\frac{1}{2} - \frac{1}{a_{k+1}a_k}\right\} < 0 \qquad a_n > \sqrt{2}$

$\therefore a_{k+2} > a_{k+1}$

$\therefore \{a_n\}$ は減少で $a_n < \sqrt{2}$

(3) $a_{n+1} - \alpha = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) - \alpha = \frac{1}{2}a_n - \left(\alpha - \frac{1}{a_n}\right) \qquad a_n > \alpha \qquad \frac{1}{a_n} < \frac{1}{\alpha} = \frac{1}{\sqrt{2}}$

$\alpha - \frac{1}{a_n} > \alpha - \frac{1}{\alpha} = \frac{\alpha^2 - 1}{\alpha} = \frac{1}{\alpha} = \frac{\alpha}{2} \qquad \alpha = \sqrt{2}$ より

$\therefore a_{n+1} - \alpha < \frac{1}{2}a_n - \frac{\alpha}{2} = \frac{1}{2}\left(a_n - \alpha\right)$

$\therefore a_{n+1} - \alpha < \frac{1}{2}\left(a_n - \alpha\right)$

3.5 $\{a_n\} \ \{b_n\} \quad a_1 = 1, b_1 = 1 \quad a_{n+1} = 2a_n - 3b_n$

$b_{n+1} = \frac{1}{2}a_n - \frac{1}{2}b_n$

(1) $\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix}$

$\therefore x_{n+1} = Ax_n \qquad \therefore x_n = A^{n-1}x_1 \qquad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(2) $\begin{vmatrix} 2-\lambda & -3 \\ \frac{1}{2} & -\frac{1}{2}-\lambda \end{vmatrix} = 0 \quad (\lambda - 2)\left(\lambda + \frac{1}{2}\right) + \frac{3}{2} = 0 \quad \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0$

$2\lambda^2 - 3\lambda + 1 = 0 \quad (2\lambda - 1)(\lambda - 1) = 0$

固有値 $\frac{1}{2}, 1$

$\lambda=\frac{1}{2}$ の固有ベクトル $\begin{pmatrix} \frac{3}{2} & -3 \\ \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $p-2q=0$ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda=1$ の固有ベクトル $\begin{pmatrix} 1 & -3 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $p-3q=0$ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\therefore \lambda=\frac{1}{2}, 1$ ${}^t(2, 1)$ ${}^t(3, 1)$

(3) $P=\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ のとき $P^{-1}=-\begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix}=\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

$P^{-1}AP=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}=D$

(4) (3)より $A=PDP^{-1}$ $\therefore A^{n-1}=PD^{n-1}P^{-1}$

$\therefore A^{n-1}=\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (\frac{1}{2})^{n-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}=\begin{pmatrix} \frac{-1}{2^{n-2}}+3 & -6 \\ -\frac{1}{2^{n-1}}+1 & \frac{3}{2^{n-1}}-2 \end{pmatrix}$

(5) (1) (4) より

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix}=\begin{pmatrix} -\frac{1}{2^{n-2}}+3 & -6 \\ -\frac{1}{2^{n-1}}+1 & \frac{3}{2^{n-1}}-2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} -\frac{1}{2^{n-2}}-3 \\ \frac{2}{2^{n-1}}-1 \end{pmatrix}$$

$\therefore a_n=-\frac{1}{2^{n-2}}-3$ $b_n=\frac{2}{2^{n-1}}-1$

$\therefore \lim_{n\to\infty} a_n=-3$ $\lim_{n\to\infty} b_n=-1$

6 (1) $f_2=1$ $f_3=2$ $f_4=3$ $f_5=5$

(2) $f_n=f_{n-1}+f_{n-2}$ $\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}=\begin{pmatrix} f_{n-1}+f_{n-2} \\ f_{n-1} \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix}$

$\therefore A=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(3) $P=\begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{pmatrix}$, $P^{-1}=\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\frac{1}{2}(1-\sqrt{5}) \\ -1 & \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}$

(4) $P^{-1}AP=\begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\frac{1}{2}(1-\sqrt{5}) \\ -1 & \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}$

$$=\begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}$$

(5) $A^{n-2}=P\begin{pmatrix} (\frac{1}{2}(1+\sqrt{5}))^{n-2} & 0 \\ 0 & (\frac{1}{2}(1-\sqrt{5}))^{n} \end{pmatrix}P^{-1}=\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \{\frac{1}{2}(1+\sqrt{5})\}^{n-2} & 0 \\ 0 & \{\frac{1}{2}(1-\sqrt{5})\}^{n-2} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

$$=\frac{1}{\sqrt{5}}\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} & -\frac{1-\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}+\frac{1+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \end{pmatrix}$$

$$\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = A^{n-2} \begin{pmatrix} f_2 \\ f_1 \end{pmatrix} = A^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore f_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \right\}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{\sqrt{5}+1}{2}+1\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{6+2\sqrt{5}}{4}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} = \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}+1\right) = \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{6-2\sqrt{5}}{4}\right) = \left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

$$\therefore f_n = \frac{1}{2^n\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n} \right\}$$

(1) $$\int_0^n |e^{ax}\sin\pi x|dx = \int_0^1 e^{ax}\sin\pi x\,dx - \int_1^2 e^{ax}\sin\pi x\,dx + \int_2^3 e^{ax}\sin\pi x\,dx \cdots + (-1)^{n-1}\int_{n-1}^n e^{ax}\sin\pi x\,dx$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \left[\frac{e^{ax}}{a^2+\pi^2}(a\sin\pi x - \pi\cos\pi x)\right]_{k-1}^{k}$$

$$= \sum_{k=1}^{n} \frac{1}{a^2+\pi^2}\left\{(-1)^{k-1+k+1}\pi\, e^{ka} + (-1)^{k-1+k+1}\pi e^{(k-1)a}\right\}$$

$$= \frac{\pi}{a^2+\pi^2}\sum_{k=1}^{n}\left(e^{ka} + e^{a(k-1)}\right) = \frac{\pi}{a^2+\pi^2}\left\{\frac{e^a(e^{na}-1)}{e^a-1} + \frac{e^{na}-1}{e^a-1}\right\}$$

$$= \frac{\pi}{a^2+\pi^2}\cdot\frac{(e^a+1)(e^{na}-1)}{e^a-1} \qquad a \neq 0$$

$$\left(\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \right)$$

(2) $\lim A_n$ が存在するためには $e^a < 1$ ∴ $a < 0$

$$\lim_{n\to\infty} A_n = -\frac{\pi(e^a+1)}{(a^2+\pi^2)(e^a-1)}$$

§4 極限値

(1) $0 \leq \dfrac{\log n}{n!} < \dfrac{n}{n!} = \dfrac{1}{(n-1)!}$

$\therefore \lim_{n\to\infty} \dfrac{\log n}{n!} = 0$

(2) $\dfrac{a^n}{n!}$ $n_0-1 < a \leqq n_0$ のとき $n_0 < n$ に対して $\dfrac{a}{n} < 1$

$\therefore \dfrac{a^n}{n!} = \dfrac{a^{n_0}}{1\cdot 2\cdots n_0}\dfrac{a}{n_0+1}\cdot\dfrac{a}{n_0+2}\cdots\cdot\dfrac{a}{n_0+(n-n_0)}$

$< \dfrac{a^{n_0}}{n_0!}\left(\dfrac{a}{n_0+1}\right)^{n-n_0}$ $\left(\dfrac{a}{n_0+1}\right)^{n-n_0} \to 0$ $n\to\infty$

$\therefore \lim_{n\to\infty}\dfrac{a^n}{n!} = 0$

(3) $a_n = \dfrac{n!}{n^n}$ とおくと $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{(n+1)^{n+1}}\,\dfrac{n^n}{n!} = \left(\dfrac{n}{n+1}\right)^n = \dfrac{1}{(1+\frac{1}{n})^n}$

$\therefore \lim_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty}\dfrac{1}{(1+\frac{1}{n})^n} = \dfrac{1}{e} < 1$

$\therefore$ $\dfrac{1}{e} < r < 1$ とすると

$\exists n_0$ $\forall n \geq n_0$ に対して $\dfrac{a_{n+1}}{a_n} < r$ $\therefore a_{n+1} < a_n r$

$\therefore$ $a_{n+m} < a_n r^m$ $\lim_{m\to\infty} r^m = 0$

$\therefore \lim_{n\to\infty} a_n = \lim_{m\to\infty} a_{n+m} = 0$

(4) $\lim_{n\to\infty}(\log(n+1) - \log n) = \lim_{n\to\infty}\log\left(1+\dfrac{1}{n}\right) = 0$

(5) $\lim_{n\to\infty}\sqrt{n}(\sqrt{n+3}-\sqrt{n-2}) = \lim_{n\to\infty}\dfrac{\sqrt{n}\{n+3-(n-2)\}}{\sqrt{n+3}+\sqrt{n-2}}$

$= \lim_{n\to\infty}\dfrac{5\sqrt{n}}{\sqrt{n+3}+\sqrt{n-2}} = \lim_{n\to\infty}\dfrac{5}{\sqrt{1+\frac{3}{n}}+\sqrt{1-\frac{2}{n}}} = \dfrac{5}{2}$

(6) $\lim_{n\to\infty}\left(1+\dfrac{1}{2n}\right)^n = \lim_{n\to\infty}\left\{\left(1+\dfrac{1}{2n}\right)^{2n}\right\}^{\frac{1}{2}} = \sqrt{e}$

(7) $\lim_{n\to\infty}\dfrac{3n+1}{n^2} = \lim_{n\to\infty}\left(\dfrac{3}{n}+\dfrac{1}{n^2}\right) = 0$

(8) $\lim_{n\to\infty}\left(1-\dfrac{2}{n}\right)^n = \lim_{n\to\infty}\left(\dfrac{n-2}{n}\right)^n = \lim_{n\to\infty}\dfrac{1}{\left(\frac{n}{n-2}\right)^n} = \lim_{n\to\infty}\dfrac{1}{\left(1+\frac{2}{n-2}\right)^n}$

$= \lim_{n\to\infty}\dfrac{1}{\left\{\left(1+\frac{2}{n-2}\right)^{\frac{n-2}{2}}\right\}^2\left(1+\frac{2}{n-2}\right)^2} = \dfrac{1}{e^2}$

(9) $\lim\limits_{n\to\infty}(n-\sqrt{n^2+3n}) = \lim\limits_{n\to\infty}\dfrac{n^2-n^2-3n}{n+\sqrt{n^2+3n}} = \lim\limits_{n\to\infty}\dfrac{-3}{1+\sqrt{1+\frac{3}{n}}} = -\dfrac{3}{2}$

(10) $\lim\limits_{n\to\infty}\sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\frac{1}{n}}+1} = \dfrac{1}{2}$

2 (1) $\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{x^3} = \lim\limits_{x\to 0}\dfrac{\cos x-\sec^2 x}{3x^2} = \lim\limits_{x\to 0}\dfrac{\cos^3 x-1}{3x^2\cos^2 x}$

$= \lim\limits_{x\to 0}\dfrac{-3\cos^2 x\sin x}{6x\cos^2 x-6x^2\sin x\cos x} = \lim\limits_{x\to 0}\dfrac{-3\cos^3 x+6\cos x\sin^2 x}{6\cos^2 x-24x\sin x\cos x-6x^2(\cos^2 x-\sin^2 x)}$

$= -\dfrac{3}{6} = -\dfrac{1}{2}$

(2) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x} = \lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}\cdot 5 = 5$

(3) $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$

(4) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x} = \lim\limits_{x\to 0}\dfrac{2e^{2x}}{1} = 2$

(5) $\lim\limits_{x\to 0}\dfrac{x^2+\sqrt{(1-\cos^2 x)x^2+2x^5}}{x^2(x+1)} = \lim\limits_{x\to 0}\dfrac{1+\sqrt{\frac{1}{x^2}\sin^2 x+2x}}{x+1} = \dfrac{1+1}{2} = 1$

(6) (1)と同じ

$\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3} = \dfrac{1}{2}$

(7) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2x+3x^2}-\sqrt{1-3x}}{3x} = \lim\limits_{x\to 0}\dfrac{5x+3x^2}{3x(\sqrt{1+2x+3x^2}+\sqrt{1-3x})} = \dfrac{5}{3\cdot 2} = \dfrac{5}{6}$

(8) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \lim\limits_{x\to 0}\dfrac{\sin x}{2x} = \dfrac{1}{2}$

(9) $\lim\limits_{x\to 0}\dfrac{\sin(a+x)-\sin(a-x)}{x} = \lim\limits_{x\to 0}\dfrac{\cos(a+x)+\cos(a-x)}{1} = 2\cos a$

(10) $\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right) = \lim\limits_{x\to 0}\dfrac{\sin x-x}{x^2\sin x} = \lim\limits_{x\to 0}\dfrac{\cos x-1}{2x\sin x+x^2\cos x}$

$= \lim\limits_{x\to 0}\dfrac{-\sin x}{2\sin x+4x\cos x-x^2\sin x}$

$= \lim\limits_{x\to 0}\dfrac{-\sin x/x}{2\sin x/x+4\cos x-x\sin x} = \dfrac{-1}{6}$

(1) $\lim_{x\to0}\frac{x\cos x - x}{\log(1+x^3)} = \lim_{x\to0}\frac{\cos x - 1 - x\sin x}{\frac{3x^2}{1+x^3}} = \lim_{x\to0}\frac{(\cos x - 1 - x\sin x)(1+x^3)}{3x^2}$

$= \lim_{x\to0}\frac{(-2\sin^2\frac{x}{2} - x\sin x)}{3x^2}(1+x^3)$

$= \lim_{x\to0}\left(-\frac{2\sin^2\frac{x}{2}}{3x^2} - \frac{x\sin x}{3x^2}\right)(1+x^3) = -\frac{1}{6} - \frac{1}{3} = -\frac{3}{6} = -\frac{1}{2}$

(2) $y = \{\log(1+x)\}^x$ とおくと $\log y = x\log(1+x)$

$\lim_{x\to0} x\log(1+x) = 0.$ $\therefore \log y = 0$ $y = 1$

$\therefore \lim_{x\to0}\{\log(1+x)\}^x = 1$

(3) $\lim_{x\to0}\frac{x - \log(1+x)}{x^2} = \lim_{x\to0}\frac{1-\frac{1}{1+x}}{2x} = \lim_{x\to0}\frac{\frac{1}{(1+x)^2}}{2} = \frac{1}{2}$

(4) $\lim_{x\to0}\frac{x^\alpha}{e^x-1} = \lim_{x\to0}\frac{\alpha x^{\alpha-1}}{e^x}\begin{cases} = \alpha & \alpha = 1 \\ = 0 & \alpha > 1 \\ \infty & \alpha < 1 \end{cases}$ $\alpha \neq 0$

(5) $\lim_{x\to0}\frac{a^x - b^x}{x} = \lim_{x\to0}\frac{a^x\log a - b^x\log b}{1} = \log a - \log b$

(6) $\lim_{x\to0}\frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to0}\frac{e^x + e^{-x} - 2}{1-\cos x} = \lim_{x\to0}\frac{e^x - e^{-x}}{\sin x} = \lim_{x\to0}\frac{e^x + e^{-x}}{\cos x} = 2$

(7) $\lim_{x\to0}(1+ax)^{\frac{1}{x}} = \lim_{x\to0}\{(1+ax)^{\frac{1}{ax}}\}^a = e^a$

(8) $\lim_{x\to0}\frac{3^x - 1}{x} = \lim_{x\to0} 3^x\log 3 = \log 3$

(9) $\lim_{x\to0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x\to0}\frac{-\sin x + x}{4x^3} = \lim_{x\to0}\frac{1-\cos x}{12x^2} = \lim_{x\to0}\frac{\sin x}{24x} = \frac{1}{24}$

(10) $y = \left(\frac{1+x}{1-x}\right)^{\frac{1}{x}}$ とおくと $\log y = \frac{1}{x}\{\log(1+x) - \log(1-x)\}$

$\lim_{x\to0}\frac{1}{x}\{\log(1+x) - \log(1-x)\} = \lim_{x\to0}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = 2$

$\therefore \lim_{x\to0}\left(\frac{1+x}{1-x}\right)^{\frac{1}{x}} = e^2$

(11) $\lim_{x\to0}\frac{x}{\sqrt{1+\tan^{-1}2x} - \sqrt{1+\tan^{-1}x}} = \lim_{x\to0}\frac{x(\sqrt{1+\tan^{-1}2x} + \sqrt{1+\tan^{-1}x})}{\tan^{-1}2x - \tan^{-1}x}$

$= \lim_{x\to0}\frac{\sqrt{1+\tan^{-1}2x} + \sqrt{1+\tan^{-1}x} - x\{\frac{1}{\sqrt{1+\tan^{-1}2x}(1+4x^2)} + \frac{1}{2\sqrt{1+\tan^{-1}x}(1+x^2)}}{\frac{2}{1+(2x)^2} - \frac{1}{1+x^2}}$

$= \frac{2}{1} = 2$

(12) $\lim_{x\to 0} \dfrac{e^x-1-x}{x^2} = \lim_{x\to 0} \dfrac{e^x-1}{2x} = \lim_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

4.4 (1) $\left(\dfrac{1}{x}\right)^{\tan x} = y$ とおく $\log y = -\tan x \log x = -\dfrac{\sin x \log x}{\cos x}$

$\lim_{x\to 0} \sin x \log x = \lim_{x\to 0} \dfrac{\log x}{\frac{1}{\sin x}} = \lim_{x\to 0} \dfrac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0} -\dfrac{\sin^2 x}{x\cos x} = 0$

$\therefore \lim_{x\to 0} \log y = 0$. $\therefore \lim_{x\to 0} \left(\dfrac{1}{x}\right)^{\tan x} = e^0 = 1$

(2) $(1-\sin 2x)^{\frac{1}{x}} = y$ とおく $\log y = \dfrac{1}{x}\log(1-\sin 2x)$

$\lim_{x\to 0} \dfrac{1}{x}\log(1-\sin 2x) = \lim_{x\to 0} \dfrac{1}{1}\left(\dfrac{-2\cos 2x}{1-\sin 2x}\right) = -2$

$\therefore \lim_{x\to 0} \log y = -2$ $\therefore \lim_{x\to 0} (1-\sin 2x)^{\frac{1}{x}} = e^{-2}$

(3) $\lim_{x\to 0} x\log x = \lim_{x\to 0} \dfrac{\log x}{\frac{1}{x}} = \lim_{x\to 0} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0}(-x) = 0$

(4) $x^x = y$ とおく $\log y = x\log x$

$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} x\log x = 0$

$\therefore \lim_{x\to 0} x^x = e^0 = 1$

(5) $(\sin x)^x = y$ とおく $\log y = x\log(\sin x)$

$\lim_{x\to 0} x\log(\sin x) = \lim_{x\to 0} \dfrac{\log(\sin x)}{\frac{1}{x}} = \lim_{x\to 0} \dfrac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = 0$

$\therefore \lim_{x\to 0} (\sin x)^x = 1$

4.5 (1) $\lim_{x\to 1} \dfrac{x^n-1}{x-1} = \lim_{x\to 1} \dfrac{nx^{n-1}}{1} = n$

(2) $\lim_{x\to 1} \dfrac{x^2+2x-3}{x-1} = \lim_{x\to 1} \dfrac{2x+2}{1} = 4$

(3) $\sqrt{x^{\frac{1}{x-1}}} = y$ $y = x^{\frac{1}{2(x-1)}}$ $\log y = \dfrac{1}{2(x-1)}\log x$

$\lim_{x\to 1} \dfrac{\log x}{2(x-1)} = \lim_{x\to 1} \dfrac{\frac{1}{x}}{2} = \dfrac{1}{2}$

$\therefore \lim_{x\to 1} x^{\frac{1}{2(x-1)}} = e^{\frac{1}{2}}$

4.6 (1) $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}=\frac{1}{2}$

(2) $\lim_{x\to\infty}\frac{(x+a)\log(x+a)}{x\log x}=\lim_{x\to\infty}\frac{1+\log(x+a)}{1+\log x}=\lim_{x\to\infty}\frac{\frac{1}{x+a}}{\frac{1}{x}}=1$

(3) $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x=\lim_{x\to\infty}\frac{1}{\left(1-\frac{1}{x}\right)^{-x}}=\frac{1}{e}$

(4) $\lim_{x\to\infty}\frac{x^2-\sin^2 x}{x^4}=\lim_{x\to\infty}\left(\frac{1}{x^2}-\frac{\sin^2 x}{x^4}\right)=0$

(5) $x^{\frac{1}{x}}=y$ とおくと $\log y=\frac{1}{x}\log x$

$\lim_{x\to\infty}\frac{1}{x}\log x=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$

$\therefore \lim_{x\to\infty}x^{\frac{1}{x}}=1$

(6) $\lim_{x\to\infty}x\left(\tan^{-1}x-\frac{\pi}{2}\right)=\lim_{x\to\infty}\frac{\tan^{-1}x-\frac{\pi}{2}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}}=-1$

(7) $\left(\frac{\pi}{2}-\tan^{-1}x\right)^{\frac{1}{x}}=y$ とおくと $\log y=\frac{1}{x}\log\left(\frac{\pi}{2}-\tan^{-1}x\right)$

$\lim_{x\to\infty}\frac{1}{x}\log\left(\frac{\pi}{2}-\tan^{-1}x\right)=\lim_{x\to\infty}\frac{1}{\frac{\pi}{2}-\tan^{-1}x}\cdot\frac{-1}{1+x^2}=0$

$\left(\lim_{x\to\infty}(1+x^2)\left(\frac{\pi}{2}-\tan^{-1}x\right)=\lim_{x\to\infty}\frac{1+x^2}{x}\,x\left(\frac{\pi}{2}-\tan^{-1}x\right)=\infty\right)$

$\therefore \lim_{x\to\infty}\left(\frac{\pi}{2}-\tan^{-1}x\right)^{\frac{1}{x}}=1$

(8) $\lim_{x\to\infty}\frac{x^2}{e^{2x}}=\lim_{x\to\infty}\frac{2}{4e^{2x}}=0$

(9) $\lim_{x\to\infty}\frac{\log_a(1+x)}{x}=\lim_{x\to\infty}\frac{1}{1}\,\frac{\frac{1}{1+x}}{\log a}=0$

(10) $\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{3!}{e^x}=0$

(11) $\lim_{x\to\infty}\frac{x^3}{e^{ax}}=\lim_{x\to\infty}\frac{3!}{a^3e^{ax}}=0 \quad a>0$

(12) $\lim_{x\to\infty}\frac{\log(2x+1)}{\log(x+1)}=\lim_{x\to\infty}\frac{\frac{2}{2x+1}}{\frac{1}{x+1}}=\lim_{x\to\infty}\frac{2(x+1)}{2x+1}=1$

(13) $\lim_{x\to\infty}x\left(\sqrt{x^2+4}-x\right)=\lim_{x\to\infty}x\,\frac{4}{\sqrt{x^2+4}+x}=2$

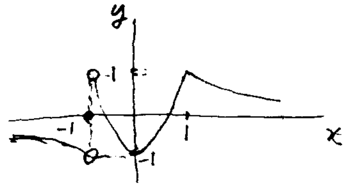
(14) $\lim_{x\to\infty}\frac{x}{a^x}=\lim_{x\to\infty}\frac{1}{a^x\log a}=\begin{cases}0 & a>1\\ -\infty & 0<a<1\end{cases}$

4.7 $\displaystyle\lim_{\theta\to\frac{\pi}{2}-0}(\pi-2\theta)\tan\theta$ $\qquad \frac{\pi}{2}-\theta=t$ とおく $\quad \theta\to\frac{\pi}{2}-0 \quad t\to+0$

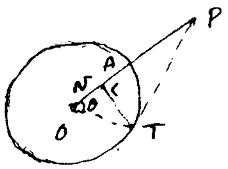
$$=\lim_{t\to+0}2t\tan\left(\frac{\pi}{2}-t\right)=\lim_{t\to+0}2t\cot t=\lim_{t\to+0}\frac{2t\cos t}{\sin t}=2$$

4.8 $\displaystyle\lim_{n\to\infty}\frac{x^{2n-1}+2x^2-1}{x^{2n}+1}=\lim_{n\to\infty}\frac{\frac{1}{x}+\frac{2}{x^{2n-2}}-\frac{1}{x^{2n}}}{1+\frac{1}{x^{2n}}}$

$$=\begin{cases}\frac{1}{x} & |x|>1\\ 0 & x=-1\\ 1 & x=1\\ 2x^2-1 & |x|<1\end{cases}$$



4.9



$\angle AOT=\theta$ とおくと

$T\to A$ のとき $\theta\to0$

$ON=r\cos\theta$

$NA=r(1-\cos\theta) \qquad OP=r\dfrac{1}{\cos\theta}$

$AP=r\left(\dfrac{1}{\cos\theta}-1\right)=r\dfrac{1}{\cos\theta}(1-\cos\theta)$

$$\lim_{T\to A}\frac{NA}{AP}=\lim_{\theta\to0}\frac{1-\cos\theta}{\frac{1}{\cos\theta}(1-\cos\theta)}=\lim_{\theta\to0}\cos\theta=1$$

4.10 $a_0=a \quad a_{n+1}=pa_n+q \qquad b_n=pa_n+\lambda$

(1) $b_{n+1}=pa_{n+1}+\lambda=p(pa_n+q)+\lambda=p^2a_n+pq+\lambda=p(b_n-\lambda)+pq+\lambda$

$\quad =pb_n+pq+\lambda(1-p)$

(2) $a_{n+1}+\alpha=p(a_n+\alpha)$ とおくと

$a_{n+1}=pa_n+\alpha(p-1) \qquad \therefore \alpha(p-1)=q \qquad \alpha=\dfrac{q}{p-1} \quad p\neq1$

$\therefore a_{n+1}+\dfrac{q}{p-1}=p\left(a_n+\dfrac{q}{p-1}\right) \qquad \left\{a_n+\dfrac{q}{p-1}\right\}$ は 公比 p 初項

$a_0+\dfrac{q}{p-1}$ の等比数列

$\therefore a_n+\dfrac{q}{p-1}=p^n\left(a_0+\dfrac{q}{p-1}\right)$

$\therefore a_n=p^n\left(a_0+\dfrac{q}{p-1}\right)-\dfrac{q}{p-1}, \qquad p\neq1$

$p=1$ のとき

$a_{n+1}-a_n=q \qquad \therefore$ 初項 a 公差 q の等差数列

$\therefore a_n=a+nq$

(3) $|p|<1$ のとき $\displaystyle\lim_{n\to\infty}a_n=-\frac{q}{p-1}$

$|p|\geq1 \quad q\neq0$ のとき 発散

4.11 (1) $(k+1)^3-k^3=3k^2+3k+1$

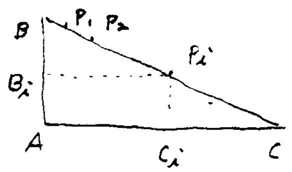
$\therefore \sum_{k=1}^{n}\{(k+1)^3-k^3\}=3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+n$

$(n+1)^3-1=3\sum_{k=1}^{n}k^2+3\frac{n(n+1)}{2}+n$

$3\sum_{k=1}^{n}k^2=n^3+3n^2+3n-\frac{3}{2}(n^2+n)-n=\frac{1}{2}(2n^3+3n^2+n)$

$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$

(2)



$BC=a$

BC を $n+1$ 等分する点を P_i $1\leqq i\leqq n$ とする

P_i より AB, AC への垂線の足を B_i, C_i とする

B_i, C_i はそれぞれ AB, AC の $n+1$ 等分点である。

$\therefore AP_i^2=(\frac{i}{n+1}AC)^2+(\frac{n+1-i}{n+1}AB)^2$

$\therefore \sum_{i=1}^{n}AP_i^2=\sum_{i=1}^{n}(\frac{i}{n+1})^2AC^2+\sum_{i=1}^{n}(\frac{n+1-i}{n+1})^2AB^2$ $n+1-i=k$ とおくと

$=\sum_{i=1}^{n}(\frac{i}{n+1})^2AC^2+\sum_{k=1}^{n}(\frac{k}{n+1})^2AB^2$

$=\frac{n(n+1)(2n+1)}{6(n+1)^2}AC^2+\frac{n(n+1)(2n+1)}{6(n+1)^2}AB^2$

$=\frac{a^2n(2n+1)}{6(n+1)}$

(3) $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}D_i=\lim_{n\to\infty}\frac{(2n+1)}{6(n+1)}a^2=\frac{1}{3}a^2$

12 (1) $\lim_{n\to\infty}(1+\frac{1}{n^2})^n=\lim_{n\to\infty}\{(1+\frac{1}{n^2})^{n^2}\}^{\frac{1}{n}}=1$

(2) $a_n=\sqrt[n]{n!}$ とおくと $(a_n)^n=n!$

$\forall m>0$ m は自然数に対して $n>m$ のとき

$n!=n(n-1)\cdots m(m-1)\cdots 2\,1>m^{n+1-m}$

$\therefore a_n>m^{\frac{n+1-m}{n}}=m^{1-\frac{m-1}{n}}$

ここで $n\to\infty$ のとき $m^{1-\frac{m-1}{n}}\to m$

$\therefore \lim_{n\to\infty}a_n\geqq m$ $\therefore \lim_{n\to\infty}a_n=\infty$

(3) $a_n = \dfrac{n}{\sqrt[n]{n!}}$ 　　$b_n = (a_n)^n = \dfrac{n^n}{n!}$ とおく

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^{n+1}}{n+1} \cdot \frac{1}{n^n} = \left(1+\frac{1}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

∴ 任意の正数 $\varepsilon > 0$ に対して十分大なる n_0 をとれば

$\forall n \geqq n_0$ に対して 　$\left|\dfrac{b_{n+1}}{b_n} - e\right| < \varepsilon$

$$\therefore e-\varepsilon < \frac{b_{n+1}}{b_n} < e+\varepsilon \quad \therefore b_n(e-\varepsilon) < b_{n+1} < b_n(e+\varepsilon)$$

$$\therefore b_{n_0}(e-\varepsilon) < b_{n_0+1} < b_{n_0}(e+\varepsilon)$$

$$\therefore b_{n_0}(e-\varepsilon)^m < b_{n_0+m} < b_{n_0}(e+\varepsilon)^m$$

$n = n_0 + m$ とおくと 　$m \to \infty$ のとき $n \to \infty$

$$b_{n_0}^{\frac{1}{n_0+m}} (e-\varepsilon)^{\frac{m}{n_0+m}} < \frac{n}{\sqrt[n]{n!}} < b_{n_0}^{\frac{1}{n_0+m}} (e+\varepsilon)^{\frac{m}{n_0+m}}$$

$$\therefore \quad e - \varepsilon \leqq \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} \leqq e + \varepsilon$$

$$\therefore \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$$

4.13 　$\lim\limits_{n\to\infty} a_n = \alpha$ 　$\forall \varepsilon > 0$ に対して十分大なる n_0 が存在して

$\forall n > n_0$ のとき 　$|a_n - \alpha| < \varepsilon$

4.14 (1) $a > 0$ 　$b > 0$ のとき

$$\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geqq 0$$

$\therefore \dfrac{a+b}{2} \geqq \sqrt{ab}$ 　等号が成り立つのは $a = b$ のときのみ

(2) $a_n = \dfrac{1 \cdot 3 \cdots\cdots (2n-1)}{2 \cdot 4 \cdots\cdots 2n}$ のとき $a_n < \dfrac{1}{\sqrt{2n+1}}$ であること

[I] $n=1$ のとき 　$a_1 = \dfrac{1}{2} < \dfrac{1}{\sqrt{3}}$ 　∴ 成り立つ

[II] $n=k$ のとき $a_k < \dfrac{1}{\sqrt{2k+1}}$ とすると

$$a_{k+1} = a_k \frac{2k+1}{2k+2} < \frac{1}{\sqrt{2k+1}} \frac{2k+1}{2(k+1)} = \frac{1}{2} \frac{\sqrt{2k+1}}{k+1} < \frac{1}{\sqrt{2n+3}}$$

$$\left(\because (\sqrt{2n+1}\sqrt{2n+3})^2 - (2(k+1))^2 = -1 < 0 \quad \therefore \sqrt{2n+1}\sqrt{2n+3} < 2(k+1)\right)$$

[I].[II]より すべての n について $a_n < \frac{1}{\sqrt{2n+1}}$

(3) ∴ $0 \leqq \lim_{n\to\infty} a_n \leqq \lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$ ∴ $\lim_{n\to\infty} a_n = 0$

4.15 $S_n = \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{n+1}{2n}$ ∴ $\lim_{n\to\infty} S_n = \frac{1}{2}$

$T_n = \sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n} - S_n\right)^2} = \sqrt{\frac{1}{n}\left\{\frac{1}{n^2}\sum k^2 - 2S_n\frac{1}{n}\sum k + nS_n^2\right\}}$

$= \sqrt{\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} - 2\frac{n+1}{2n}\frac{1}{n^2}\frac{n(n+1)}{2} + \frac{(n+1)^2}{4n^2}}$

$= \sqrt{\frac{(n+1)(2n+1)}{6n^2} - \frac{(n+1)^2}{2n^2} + \left(\frac{n+1}{2n}\right)^2}$

∴ $\lim_{n\to\infty} T_n = \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$

4.16 $f(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k!}$ $f(n,0) = 1$

(1) $f(n,k) + f(n,k+1) = \frac{n(n-1)\cdots(n-k+1)}{k!} + \frac{n(n-1)\cdots(n-k+1)(n-k)}{(k+1)!}$

$= \frac{(k+1+n-k)n(n-1)\cdots(n-k+1)}{(k+1)!}$

$= \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!} = f(n+1,k+1)$

(2) $\sum_{k=0}^{n} f(n,k) = 2^n$ であること

[I] $n=1$ のとき

$\sum_{k=0}^{1} f(1,k) = f(1,0) + f(1,1) = 1+1 = 2$.

∴ 成り立つ

[II] $n=m$ のとき成り立つとすると $\sum_{k=0}^{m} f(m,k) = 2^m$

$n=m+1$ のとき

$\sum_{k=0}^{m+1} f(m+1,k) = \sum_{k=1}^{m}\{f(m,k-1) + f(m,k)\} + f(m+1,m+1)$

$= 2^m + 2^m - 1 + 1 = 2^{m+1}$ ((1)より)

∴ $n=m+1$ のとき成り立つ

[I].[II]より (2)は成り立つ

(3) (2)より $n > m$ のとき $2^n > f(n, m+1) = \dfrac{n(n-1)(n-2)\cdots(n-m)}{(m+1)!}$

$$\therefore\ 0 < \frac{n^m}{2^n} < \frac{n^m (m+1)!}{n(n-1)(n-2)\cdots(n-m)}$$

$$= \frac{(m+1)!}{n(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m}{n})}$$

$$\therefore\ 0 \leqq \lim_{n\to\infty} \frac{n^m}{2^n} \leqq 0 \quad \therefore\ \lim_{n\to\infty} \frac{n^m}{2^n} = 0$$

4.17 $\cos\theta + \sin\theta = \dfrac{1}{\sqrt{2}}$ $\therefore\ 1 + 2\sin\theta\cos\theta = \dfrac{1}{2}$

$\therefore\ \sin 2\theta = -\dfrac{1}{2}$

$$\therefore\ \sum_{i=0}^{\infty} (\sin 2\theta)^i = \sum_{i=0}^{\infty} \left(-\frac{1}{2}\right)^i = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$$

§5 連続性と微分可能

5.1 (1) $f(x)=\begin{cases}\dfrac{x(e^{\frac{1}{x}}+e^{-\frac{1}{x}})}{e^{\frac{1}{x}}-e^{-\frac{1}{x}}} & x\neq 0\\ 0 & x=0\end{cases}$

$x\to+0$ のとき $f(x)\to 0$. $x\to-0$ のとき $f(x)\to 0$

$\therefore$ $f(x)$ は $x=0$ で連続

$\dfrac{1}{x}(f(x)-f(0))=\dfrac{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}$

$\therefore$ $f_+'(0)=1$ $\quad f_-'(0)=-1$

$\therefore$ $f(x)$ は $x=0$ で微分不可能

(2) $f(x)=\begin{cases}x^h\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}$

$h>1$ のとき $f'(0)=0$ $x=0$ で微分可能，連続

$h=1$ のとき $x=0$ で微分不可能 連続

5.2 $f(x)=\begin{cases}x^2\sin\dfrac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$

(1) $x\neq 0$ のとき $f'(x)=2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$

$x=0$ のとき $f'(0)=0$

(2) $\displaystyle\lim_{x\to 0}f'(x)$ は存在しない．$\therefore$ $f'(x)$ は $x=0$ で不連続

5.3 (1) $f(x)$ が $x=a$ で連続

$\forall\varepsilon>0$. $\exists\delta>0$ $\quad |x-a|<\delta$ のとき $|f(x)-f(a)|<\varepsilon$

(2) $\forall\varepsilon>0$ $\exists A$, $\exists\delta>0$ $|x-a|<\delta$ のとき $\left|\dfrac{f(x)-f(a)}{x-a}-A\right|<\varepsilon$

5.4 $F(x)=f(x)-f(a)-\dfrac{f(b)-f(a)}{b-a}(x-a)$ とおくと

$F(a)=F(b)=0$

$a\leqq x\leqq b$ で $F(x)=0$ ならば $F'(c)=0$ $\therefore$ $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ $\quad a<c<b$

$F(x)\neq 0$ ならば $F(x)$ は $x=c$ で最大値(または最小値)をとる

$F_+'(c)\leqq 0$, $F_-'(c)\geqq 0$ (または $F_+'(c)\geqq 0$, $F_-'(c)\leqq 0$)

$F_+'(c)=F_-'(c)$ より $F'(c)=0$ $\therefore$ $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ $\quad a<c<b$.

5.5 (1) $f_n(x) = nx(1-x)^n \quad [0,1]$

$0<p<1$ のとき $\frac{1}{q}=p$ とおくと $q>1$

$np^n = \frac{n}{q^n} \qquad \lim_{x\to\infty}\frac{x}{q^x} = \lim_{x\to\infty}\frac{1}{q^x \log q} = 0$

$\therefore \lim_{n\to\infty} np^n = 0$

$0<x<1$ のとき $p=1-x$ とおくと $0<p<1$

$\therefore \lim_{n\to\infty} nx(1-x)^n = 0 \qquad x=0$ or 1 のとき $f_n(x)=0$

$\therefore \lim_{n\to\infty} f_n(x) = 0$

(2) $\lim_{n\to\infty} f_n(x) = f(x)$ とおくと $f(x)=0$

$\therefore \forall \varepsilon>0$ に対して

$|f_n(x) - f(x)| < \varepsilon \quad \therefore nx(1-x)^n < \varepsilon$

$x \to 1-0$ のとき $x(1-x)^n$ は 0 に近づき、

$x=\frac{1}{2}$ のとき $\frac{n}{2^{n+1}} < \varepsilon$ となるのは n が十分大なるとき

$nx(1-x)^n < \varepsilon$ を満たす n は x の値によって異なる

一様ではない。

§ 6 微分

6.1 (1) $y = e^{-\frac{1}{x^2}}$ $\qquad y' = \frac{2}{x^3} e^{-\frac{1}{x^2}}$

(2) $y = \log|x+\sqrt{x^2+A}|$ $\qquad y' = \frac{1}{x+\sqrt{x^2+A}}\left(1+\frac{x}{\sqrt{x^2+A}}\right) = \frac{1}{\sqrt{x^2+A}}$

(3) $y = x\cos x$ $\qquad y' = \cos x - x\sin x$

(4) $y = x\log(x^2+1)$ $\qquad y' = \log(x^2+1) + \frac{2x^2}{x^2+1}$

(5) $y = \frac{a^x}{\sqrt{x^2+a^2}}$ $\qquad y' = \frac{a^x \log a}{\sqrt{a^2+x^2}} + a^x\left(-\frac{1}{2}\right)(x^2+a^2)^{-\frac{3}{2}}\cdot 2x$

$$= \frac{a^x\log a}{\sqrt{x^2+a^2}} - \frac{x a^x}{(\sqrt{x^2+a^2})^3}$$

(6) $y = \log\frac{x-1}{x+1}$ $\qquad y' = \frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$

(7) $y = \log\cos x$ $\qquad y' = \frac{-\sin x}{\cos x} = -\tan x$

(8) $y = \sqrt{x^2+3x+5}$ $\qquad y' = \frac{2x+3}{2\sqrt{x^2+3x+5}}$

(9) $y = e^{-x}\sin x$ $\qquad y' = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x - \sin x)$

(10) $y = \sin\frac{x}{a}$ $\qquad y' = \frac{1}{a}\cos\frac{x}{a}$

(11) $y = \log\frac{\sqrt{x+2}+\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+1}} = \log\frac{(\sqrt{x+2}+\sqrt{x+1})^2}{(x+2)-(x+1)} = 2\log(\sqrt{x+2}+\sqrt{x-1})$

$y' = \frac{2}{\sqrt{x+2}+\sqrt{x+1}}\left(\frac{1}{2\sqrt{x+2}} + \frac{1}{2\sqrt{x+1}}\right) = \frac{1}{\sqrt{x+2}\sqrt{x+1}}$

(12) $y = (1+x^2-x^4)^{\frac{1}{3}}$ $\qquad y' = \frac{1}{3}(1+x^2-x^4)^{-\frac{2}{3}}(2x-4x^3)$

$$= \frac{2x-4x^3}{3(\sqrt[3]{1+x^2-x^4})^2}$$

6.2 (1) $y = \frac{1}{2}\{x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\}$

$y' = \frac{1}{2}\{\sqrt{a^2-x^2} + \frac{-x^2}{\sqrt{a^2-x^2}} + a^2\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{1}{a}\} = \sqrt{a^2-x^2}$

(2) $y = \sin^{-1}5x$ $\qquad y' = \frac{5}{\sqrt{1-25x^2}}$

(3) $y = \sin^{-1}\sqrt{1-x^2}$ $\qquad y' = \frac{1}{\sqrt{1-(1-x^2)}}\,\frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}} = \frac{\mp 1}{\sqrt{1-x^2}}$ $\qquad x>0$のとき $-$, $x<0$のとき $+$

(4) $y = \tan^{-1}x + \tan^{-1}\frac{1}{x}$ $\qquad y' = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = 0$

(5) $y = \tan^{-1}ax$ $\qquad y' = \frac{a}{1+a^2x^2}$

(6) $y = x\sin^{-1}\sqrt{1-x^2}$ $\qquad y' = \sin^{-1}\sqrt{1-x^2} + x\frac{\mp 1}{\sqrt{1-x^2}} = \sin^{-1}\sqrt{1-x^2} \mp \frac{x}{\sqrt{1-x^2}}$

$x>0$のとき $-$, $x<0$のとき $+$

(7) $y=(1+x^2)\tan^{-1}x \qquad y'=2x\tan^{-1}x+\dfrac{1+x^2}{1+x^2}=2x\tan^{-1}x+1$

(8) $y=\sin^{-1}e^x \qquad y'=\dfrac{e^x}{\sqrt{1-e^{2x}}}$

(9) $y=\tan^{-1}\dfrac{\sqrt{x-1}}{\sqrt{x+1}} \qquad y'=\dfrac{1}{1+\frac{x-1}{x+1}}\left\{\dfrac{1}{2\sqrt{x+1}\sqrt{x-1}}-\dfrac{\sqrt{x-1}}{2(\sqrt{x+1})^3}\right\}$

$$=\frac{x+1}{x+1+x-1}\,\frac{1}{2}\left(\frac{1}{\sqrt{x+1}\sqrt{x-1}}-\frac{\sqrt{x-1}}{(\sqrt{x+1})^3}\right)$$

$$=\frac{1}{4x}\left(\frac{\sqrt{x+1}}{\sqrt{x-1}}-\frac{\sqrt{x-1}}{\sqrt{x+1}}\right)=\frac{1}{2x\sqrt{x^2-1}}$$

(10) $y=\tan^{-1}\left(2\tan\frac{x}{2}\right) \qquad y'=\dfrac{1}{1+4\tan^2\frac{x}{2}}\cdot 2\sec^2\frac{x}{2}\cdot\frac{1}{2}$

$$=\frac{1}{\cos^2\frac{x}{2}+4\sin^2\frac{x}{2}}=\frac{1}{1+3\sin^2\frac{x}{2}}$$

6.3 (1) $y=x^x \qquad \log y=x\log x \qquad \dfrac{y'}{y}=\log x+1$

$$y'=x^x(\log x+1)$$

(2) $y=x^{\frac{1}{x}} \qquad \log y=\frac{1}{x}\log x \qquad \dfrac{y'}{y}=\dfrac{-1}{x^2}\log x+\dfrac{1}{x^2}$

$$y'=x^{\frac{1}{x}}\frac{1}{x^2}(1-\log x)=x^{\frac{1}{x}-2}(1-\log x)$$

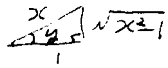
(3) $y=a^{\sqrt{x}} \qquad \log y=\sqrt{x}\log a \qquad \dfrac{y'}{y}=\dfrac{1}{2\sqrt{x}}\log a$

$$y'=\frac{a^{\sqrt{x}}}{2\sqrt{x}}\log a$$

(4) $y=\tan^{-1}x \qquad y'=\dfrac{1}{1+x^2}$

(5) $y=\sec^{-1}x \qquad x=\sec y \qquad x=\dfrac{1}{\cos y} \qquad 1=\dfrac{\sin y}{\cos^2 y}y'$

$$y'=\frac{\cos^2 y}{\sin y}=\frac{x}{\sqrt{x^2-1}}\cdot\frac{1}{x^2}=\frac{1}{x\sqrt{x^2-1}}$$



(6) $y=\log(x+\sqrt{x^2+a}) \qquad y'=\dfrac{1}{x+\sqrt{x^2+a}}\left(1+\dfrac{x}{\sqrt{x^2+a}}\right)=\dfrac{1}{\sqrt{x^2+a}}$

6.4 $\begin{cases} x=\sin^3 t \\ y=\cos^3 t\end{cases} \qquad \dfrac{dy}{dx}=\dfrac{-3\cos^2 t\sin t}{3\sin^2 t\cos t}=-\dfrac{\cos t}{\sin t}=-\cot t$

6.5 (1) $\begin{cases} x=\dfrac{1}{1+t^2} \\ y=\dfrac{2t^2}{1+t^2}\end{cases} \qquad 2x+y=2 \quad \therefore\ y=-2x+2$

$$\therefore\ y'=-2 \qquad y''=0$$

(2) $x^2y + xy^2 - 2 = 0 \qquad xy(x+y) = 2$

$2xy + y^2 + (x^2+2xy)y' = 0 \qquad y' = -\dfrac{y^2+2xy}{x^2+2xy} = -\dfrac{y(y+2x)}{x(x+2y)}$

$2y + (2x+2y)y' + (2x+2y)y' + 2xy'^2 + (x^2+2xy)y'' = 0$

$2y + 4(x+y)y' + 2xy'^2 + (x^2+2xy)y'' = 0$

$2y - 4(x+y)\dfrac{y(y+2x)}{x(x+2y)} + 2\dfrac{y^2(y+2x)^2}{x(x+2y)^2} + x(x+2y)y'' = 0$

$\dfrac{2xy(x+2y)^2 - 4y(x+y)(y+2x)(x+2y) + 2y^2(y+2x)^2}{x(x+2y)^2} = -x(x+2y)y''$

$\dfrac{-6y(x+y)(x^2+xy+y^2)}{x(x+2y)^2} = -x(x+2y)y''$

$\dfrac{12(x^2+xy+y^2)}{x^3(x+2y)^3} = y''$

6.6 $F(x) = \begin{vmatrix} f_{11}(x) & f_{12}(x) & f_{13}(x) \\ f_{21}(x) & f_{22}(x) & f_{23}(x) \\ f_{31}(x) & f_{32}(x) & f_{33}(x) \end{vmatrix}$

$F'(x) = \begin{vmatrix} f_{11}'(x) & f_{12}(x) & f_{13}(x) \\ f_{21}'(x) & f_{22}(x) & f_{23}(x) \\ f_{31}'(x) & f_{32}(x) & f_{33}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}'(x) & f_{13}(x) \\ f_{21}(x) & f_{22}'(x) & f_{23}(x) \\ f_{31}(x) & f_{32}'(x) & f_{33}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}(x) & f_{13}'(x) \\ f_{21}(x) & f_{22}(x) & f_{23}'(x) \\ f_{31}(x) & f_{32}(x) & f_{33}'(x) \end{vmatrix}$

6.7 $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$x \neq 0$ のとき $f'(x) = \dfrac{2}{x^3} e^{-\frac{1}{x^2}}$, $x = 0$ のとき $f'(0) = 0$

6.8 $f(x) = x^3 \qquad f(a+h) = f(a) + hf'(a+\theta h)$

$(a+h)^3 = a^3 + h \cdot 3(a+\theta h)^2$

$3a^2h + 3ah^2 + h^3 = 3h(a+\theta h)^2 \qquad \therefore (a+\theta h)^2 = \dfrac{3a^2+3ah+h^2}{3}$

$\therefore \theta h = \sqrt{\dfrac{3a^2+3ah+h^2}{3}} - a = \sqrt{a^2+ah+\dfrac{h^2}{3}} - a$

$\theta = \dfrac{a + \frac{h}{3}}{\sqrt{a^2+ah+\frac{h^2}{3}} + a}$

$\therefore \lim_{h \to 0} \theta = \dfrac{1}{2}$

6.9 $\dfrac{dy}{dx} = \dfrac{\sin t}{1-\cos t}$ $\qquad x = a(t - \sin t)$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt} = \dfrac{\sin t}{1-\cos t}$ $\qquad \dfrac{dx}{dt} = a(1-\cos t)$

$\therefore \dfrac{dy}{dt} = a \sin t$ $\qquad \therefore y = a(c - \cos t)$

6.10 $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\qquad g(x) = f(x) \tan^{-1} x$

$g'(0) = \lim_{h \to 0} \dfrac{1}{h} \{ g(h) - g(0) \} = \lim_{h \to 0} \dfrac{1}{h} e^{-\frac{1}{h^2}} \tan^{-1} h$

$= \lim_{h \to 0} \dfrac{1}{h e^{\frac{1}{h^2}}} \tan^{-1} h = 0$

$x \neq 0$ のとき

$g'(x) = \dfrac{2}{x^3} e^{-\frac{1}{x^2}} \tan^{-1} x + \dfrac{1}{1+x^2} e^{-\frac{1}{x^2}}$ $\qquad x \neq 0$

$g'(0) = 0$

$g''(0) = \lim_{h \to 0} \dfrac{1}{h} e^{-\frac{1}{h^2}} \left\{ \dfrac{2}{h^3} \tan^{-1} h + \dfrac{1}{1+h^2} \right\}$

$= \lim_{h \to 0} \dfrac{1}{h e^{\frac{1}{h^2}}} \left\{ \dfrac{2}{h^3} \tan^{-1} h + \dfrac{1}{1+h^2} \right\} = 0$

$\therefore g''(0) = 0$

6.11 $f(x) = \sin^{-1} x$ $\qquad f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad f''(x) = \dfrac{x}{(1-x^2)\sqrt{1-x^2}}$

$\therefore (1-x^2) f''(x) - x f'(x) = 0$

6.12 (1) $f(x) = \log |\tan \frac{x}{2}|$ $\qquad f'(x) = \dfrac{\sec^2 \frac{x}{2}}{\tan \frac{x}{2}} \dfrac{1}{2} = \dfrac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \dfrac{1}{\sin x}$

$f''(x) = \dfrac{-\cos x}{\sin^2 x}$

(2) $f(x) = \sin^{-1} x$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad f''(x) = \dfrac{x}{(\sqrt{1-x^2})^3}$

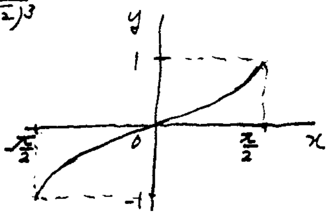
6.13 $f(x) = 2 \sin 2x \cos x$

$f'(x) = 4 \cos 2x \cos x - 2 \sin 2x \sin x$ $\qquad f'(\frac{\pi}{4}) = -\sqrt{2}$

§7. グラフ

7.1* (1) $y=\sin^{-1}x$, $y'=\dfrac{1}{\sqrt{1-x^2}}$, $y''=\dfrac{x}{(\sqrt{1-x^2})^3}$

x	-1		0		1
y'		$+$	1	$+$	
y''		$-$	0	$+$	
y	$-\frac{\pi}{2}$	↗	0	↗	$\frac{\pi}{2}$



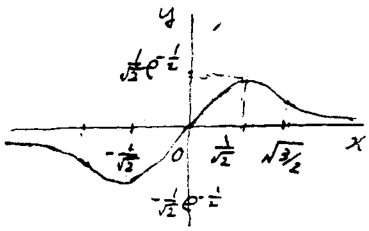
(2) $y=xe^{-x^2}$

$y'=e^{-x^2}-2x^2e^{-x^2}=(1-2x^2)e^{-x^2}$

$y''=\{-4x-2x(1-2x^2)\}e^{-x^2}$

$=-2x(3-2x^2)e^{-x^2}$

x		$-\sqrt{3/2}$		$-\sqrt{1/2}$		0		$\sqrt{1/2}$		$\sqrt{3/2}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	↘		↘	$-\frac{1}{\sqrt2}e^{-\frac12}$	↗	0	↗	$\frac{1}{\sqrt2}e^{-\frac12}$	↘		↘



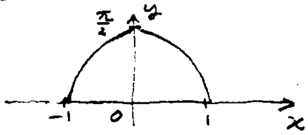
(3) $y=\sin^{-1}\sqrt{1-x^2}$

$y'=\dfrac{1}{\sqrt{1-(1-x^2)}}\ \dfrac{-x}{\sqrt{1-x^2}}=\dfrac{-x}{\sqrt{1-x^2}\,|x|}$

$y''=\dfrac{-x}{(\sqrt{1-x^2})^3}\quad x\geq 0$

$y''=\dfrac{x}{(\sqrt{1-x^2})^3}\quad x<0$

x	-1		0		1
y'		$+$		$-$	
y''		$-$		$-$	
y	0	↗	$\frac{\pi}{2}$	↘	0

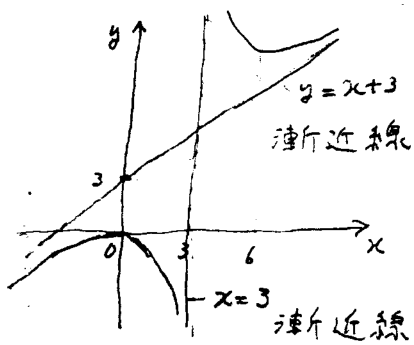


(4) $y=\dfrac{x^2}{2(x-3)}=\dfrac12\left(x+3+\dfrac{9}{x-3}\right)$

$y'=\dfrac12\left(1-\dfrac{9}{(x-3)^2}\right)=\dfrac{x(x-6)}{2(x-3)^2}$

$y''=\dfrac{9}{(x-3)^3}$

x		0		3		6	
y'	$+$	0	$-$		$-$	0	$+$
y''	$-$		$-$		$+$		$+$
y	↗		↘		↘		↗

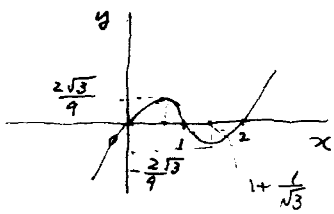


(5) $y=x(x-1)(x-2)=x^3-3x^2+2x$

$y'=3x^2-6x+2$

$y''=6(x-1)$

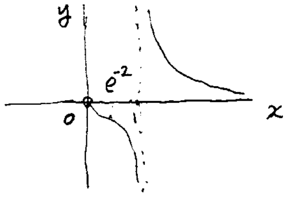
$y'=0$ より $x=\dfrac{3\pm\sqrt3}{3}$



(6) $y=\dfrac{1}{\log x}$

$y'=\dfrac{-1}{x(\log x)^2}$

$y''=\dfrac{(\log x+2)}{x^2(\log x)^3}$



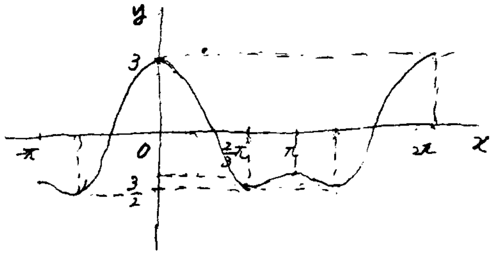
(7) $y=2\cos x+\cos 2x=2\cos^2 x+2\cos x-1$

$y'=-2\sin x-2\sin 2x$

$=-2\sin x-4\sin x\cos x$

$=-2\sin x(2\cos x+1)$

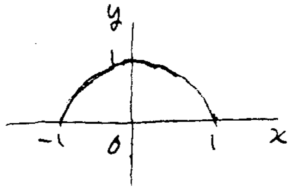
$y''=-2\cos x+4\cos 2x$



(8) $y=\sqrt{1-x^2}$

$y'=\dfrac{x}{\sqrt{1-x^2}}$

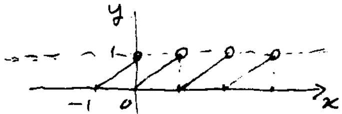
$y''=\dfrac{1}{(\sqrt{1-x^2})^3}$



(9) $y=x-[x]$

$0\leqq x<1 \quad y=x$

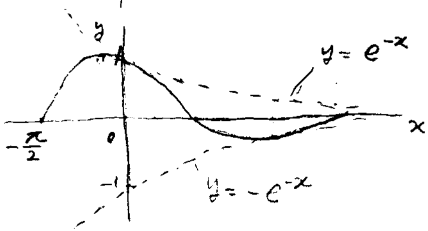
$1\leqq x<2 \quad y=x-1$



(10) $y=e^{-x}\cos x$

$y'=-(\cos x+\sin x)e^{-x}$

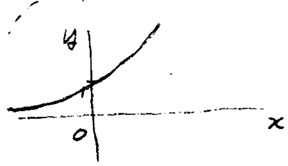
$=\sqrt{2}\,e^{-x}\cos(x+\frac{3}{4}\pi)$



(11) $y=\sinh x+\cosh x$

$=\frac{1}{2}(e^x-e^{-x})+\frac{1}{2}(e^x+e^{-x})$

$=e^x$



(12) $y=x^2e^{-x}$

$y'=(-x^2+2x)e^{-x}$

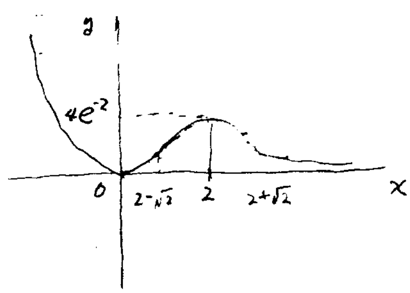
$=-x(x-2)e^{-x}$

$y''=(x^2-2x-2x+2)e^{-x}$

$=(x^2-4x+2)e^{-x}$

$x^2-4x+2=0$

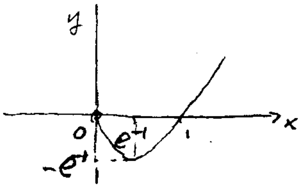
$x=2\pm\sqrt{2}$



(13) $y = x\log x$

$y' = \log x + 1$

$y'' = \frac{1}{x}$

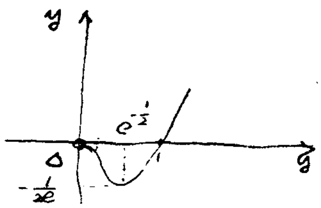


(14) $y = x^2\log x$

$y' = 2x\log x + x$

$y'' = 2\log x + 3$

$= 2(\log x + \frac{3}{2})$



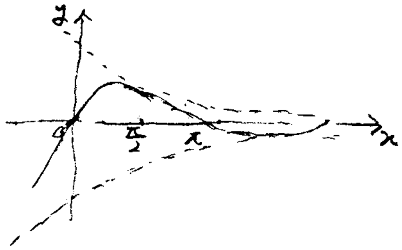
(15) $y = e^{-x}\sin x$

$y' = (-\sin x + \cos x)e^{-x}$

$= \sqrt{2}\, e^{-x}\sin(x + \frac{3}{4}\pi)\, e^{-x}$

$y'' = (-\cos x - \sin x + \sin x - \cos x)e^{-x}$

$= -2\cos x\, e^{-x}$



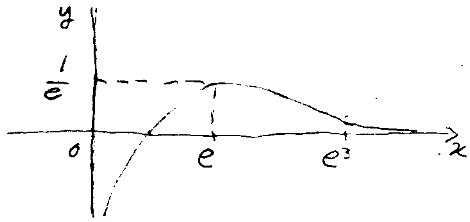
(16) $y = \frac{1}{x}\log x$

$y' = \frac{1}{x^2} - \frac{1}{x^2}\log x$

$= \frac{1}{x^2}(1 - \log x)$

$y'' = \frac{-2}{x^3}(1 - \log x) - \frac{1}{x^3}$

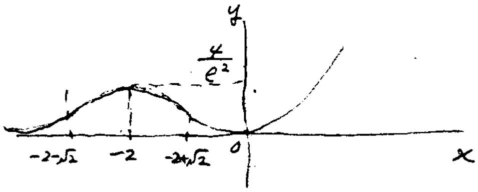
$= \frac{1}{x^3}(-3 + \log x)$



(17) $y = x^2 e^x$

$y' = (x^2 + 2x)e^x$

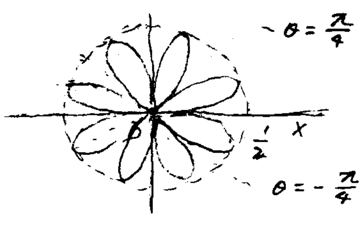
$y'' = (x^2 + 4x + 2)e^x$



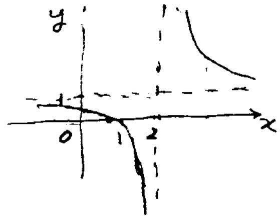
7.2 $r = |2\sin\theta\cos\theta\cos 2\theta|$

$= |\sin 2\theta\cos 2\theta|$

$= \frac{1}{2}|\sin 4\theta|$



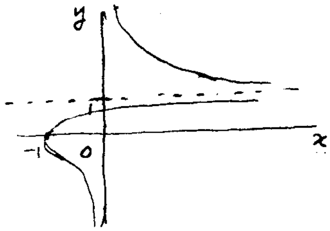
7.3 (1) $x = 1 - \frac{1}{t}$, $y = \frac{1}{t+1}$.



$x = \frac{t-1}{t}$ $\quad t(x-1) = -1$ $\quad t = \frac{1}{1-x}$

$y = \frac{1}{\frac{1}{1-x}+1} = \frac{1-x}{2-x} = 1 + \frac{1}{x-2}$

(2) $x = t^2 - 1$ $\quad y = \frac{t}{t-1}$



$y(t-1) = t$ $\quad t(y-1) = y$ $\quad t = \frac{y}{y-1}$

$x = \frac{y^2}{(y-1)^2} - 1 = \frac{2y-1}{(y-1)^2}$

$\frac{dx}{dy} = \frac{-2y}{(y-1)^3}$

7.4 (1) $y = e^x \sin x$

$y' = (\sin x + \cos x)e^x = \sqrt{2}\sin(x+\frac{\pi}{4})x$

$y'' = 2\sin(x+\frac{\pi}{2})x$

$y'=0$ より $x+\frac{\pi}{4} = 2n\pi$ $\quad x = 2n\pi - \frac{\pi}{4}$

$x = (2n+1)\pi - \frac{\pi}{4}$

$x = 2n\pi - \frac{\pi}{4}$ のとき 極小値 $\frac{-1}{\sqrt{2}} e^{(2n-\frac{1}{4})\pi}$

$x = (2n+1)\pi - \frac{\pi}{4}$ 　極大値 $\frac{1}{\sqrt{2}} e^{(2n+1-\frac{1}{4})\pi}$

(2) $y = x + \frac{1}{x-1}$

$y' = 1 - \frac{1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$

$y'' = \frac{2}{(x-1)^3}$

$x=0$ のとき 極大値 -1

$x=2$ のとき 極小値 3

7.5 $x^2+y^2 = 4x$ $\quad x + yy' = 2$ $\quad y' = 3$

$x + 3y = 2$

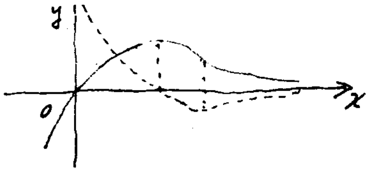
$(2-3y)^2 + y^2 = 4(2-3y)$ $\quad 10y^2 - 4 = 0$

$y = \pm\frac{2}{\sqrt{10}}$ $\quad x = 2 \mp \frac{6}{\sqrt{10}}$ $\quad (2-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}})\ (2+\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}})$

$y = 3x - 6 \pm 2\sqrt{10}$

7.6 7.4 (1) に同じ

7.7



7.8 $\begin{cases} x = \tan^{-1} t \\ y = \log |t + \sqrt{t^2+1}| \end{cases}$

$y' = \dfrac{\frac{1}{\sqrt{1+t^2}}}{\frac{1}{1+t^2}} = \sqrt{1+t^2}$

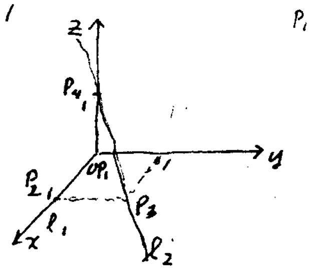
$y'' = \dfrac{t}{\sqrt{1+t^2}} \cdot \dfrac{1}{\frac{1}{1+t^2}} = t\sqrt{1+t^2}$

(0.0) が変曲点

t		0	
x	$-$	0	$+$
y'	$+$	1	$+$
y''	$-$	0	$+$
y	↗	0	↗

§8 最大・最小

8.1



$P_1(0,0,0)$ $P_2(1,0,0)$ $P_3(1,1,0)$ $P_4(0,0,1)$

l_1 ; $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

l_2 ; $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$.

l_1 が1点となり $Q(1,-1,1)$ を通る　$x-1=0$

l_2 が1点となり $Q(1,-1,1)$ を通る　$x-1+(y+1)-(z-1)=0$

$x+y-z+1=0$

l_1 を含む平面　$by+cz=0$

l_2 を含み上の平面に平行な平面　$by+c(z-1)=0$

これが P_3 を通る　∴ $b-c=0$　∴ $b=c$

それぞれ　$y+z=0$　$y+z-1=0$

これに垂直な Q を通る平面を　$a(x-1)+b(y+1)+c(z-1)=0$ とする

$a\cdot 0+b\cdot 1+c\cdot 1=0$　∴ $c=-b$

原点からの距離 $\delta=\frac{a+c-b}{\sqrt{a^2+b^2+c^2}}$　　$c=-b$ を代入

$\delta=\frac{a-2b}{\sqrt{a^2+2b^2}}$

$\frac{d\delta}{db}=-2(a^2+2b^2)^{-\frac{1}{2}}-\frac{1}{2}(a-2b)(a^2+2b^2)^{-\frac{3}{2}}4b$

$=\{-2(a^2+2b^2)-2b(a-2b)\}(a^2+2b^2)^{-\frac{3}{2}}$

$=-2a(a+b)(a^2+2b^2)^{-\frac{3}{2}}$

$a>0$ とおくと　$b<-a$　δ は増加　$b>-a$ ・減少

∴ $b=-a$ のとき最大

∴ $a(x-1)-a(y+1)+a(z-1)=0$

∴ $x-y+z-3=0$

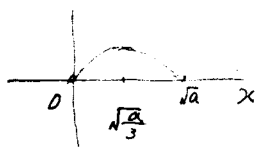
8.2　$f(x)=-x^3+ax$　$f'(x)=-3x^2+a=-3(x-\sqrt{\frac{a}{3}})(x+\sqrt{\frac{a}{3}})$　$(a>0)$

(1)　$a\leqq 0$ のとき　$M=0$　$m=-1+a$

$0<a<1$ のとき　$M=\frac{2a\sqrt{a}}{3\sqrt{3}}$　$m=-1+a$

$1\leqq a<3$ のとき　$M=\frac{2a\sqrt{a}}{3\sqrt{3}}$　$m=0$

$3\leqq a$ のとき　$M=-1+a$　$m=0$



(2)　$m=0$　$a\leqq 0$　$M=1-a$　　　$\sqrt{\frac{a}{3}}=t$ とおくと

$a<1$ のとき　$1-a-2(\sqrt{\frac{a}{3}})^3=1-3t^2-2t^3=-(1+t)^2(2t-1)$

$x < \frac{1}{2}$ のとき $1-a > 2\left(\sqrt{\frac{a}{3}}\right)^3$

$\therefore \sqrt{\frac{a}{3}} < \frac{1}{2}$ $\therefore$ $a < \frac{3}{4}$ のとき $M = 1-a$

$\frac{3}{4} \leqq a < 1$ のとき $M = 2\left(\sqrt{\frac{a}{3}}\right)^3$

$1 \leqq a < 3$ のとき $M = 2\left(\sqrt{\frac{a}{3}}\right)^3$

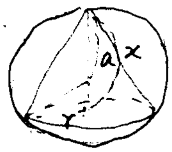
$3 \leqq a$ のとき $M = a-1$

$\therefore$ 答 $m = 0$ $a < \frac{3}{4}$ のとき $M = 1-a$

$\frac{3}{4} \leqq a < 3$ のとき $M = 2\left(\sqrt{\frac{a}{3}}\right)^3$

$3 \leqq a$ のとき $M = a-1$

8.3



$(x-a)^2 + r^2 = a^2$

$r^2 = a^2 - (x-a)^2$

$V(x) = \frac{1}{3}\pi r^2 x = \frac{\pi}{3}(2ax^2 - x^3)$

$V'(x) = \frac{\pi}{3}(4ax - 3x^2) = \frac{\pi}{3}x(4a - 3x)$

$x = \frac{4}{3}a$、$r = \frac{2\sqrt{2}}{3}a$ のとき最大値 $\frac{32\pi}{81}a^3$

8.4 $x + y = a$ $y = a - x$

$V = \frac{4}{3}\pi(x^3 + y^3) = \frac{4}{3}\pi a(x^2 - xy + y^2)$

$= \frac{4\pi a}{3}(3x^2 - 3ax + a^2)$

$V' = \frac{4\pi a}{3}(6x - 3a)$ $x = \frac{a}{2}$ のとき最小値

$V = \frac{\pi}{3}a^3$

8.5 $x^2 + xy + y^2 = 1$

(1) $2x + y + (x + 2y)\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{y + 2x}{x + 2y}$

(2) $z = xy$ $\frac{dz}{dx} = y + x\frac{dy}{dx} = y - \frac{xy + 2x^2}{x + 2y}$

$= \frac{2(y^2 - x^2)}{x + 2y}$

(3) $\frac{dz}{dx} = 0$ のとき $y = \pm x$ $\therefore$ $3x^2 = 1$ $x = \pm\frac{1}{\sqrt{3}}$ $y = \pm\frac{1}{\sqrt{3}}$

$x^2 = 1$ $x = \pm 1$ $y = \mp 1$

$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ のとき最大値 $\frac{1}{3}$

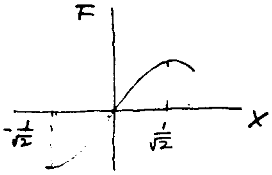
$(\pm 1, \mp 1)$ のとき最小値 -1

8.6 (1) $f(x) = \sin 3x + 3\sin x$

$= 3\sin x - 4\sin^3 x + 3\sin x$

$= 6\sin x - 4\sin^3 x$

(2) $-\frac{\pi}{6} \leqq x \leqq \frac{\pi}{3}$ $\quad -\frac{1}{2} \leqq \sin x \leqq \frac{\sqrt{3}}{2}$



$F(x) = 6x - 4x^3$

$F'(x) = 6 - 12x^2 = 6(1-\sqrt{2}x)(1+\sqrt{2}x)$

最大値 $F(\frac{1}{\sqrt{2}}) = \frac{6}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 2\sqrt{2}$ $\quad f(\frac{\pi}{4}) = 2\sqrt{2}$

最小値 $F(-\frac{1}{2}) = -3 + \frac{1}{2} = -\frac{5}{2}$ $\quad f(-\frac{\pi}{6}) = -\frac{5}{2}$

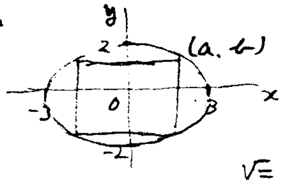
8.7 $0 \leqq \theta \leqq \frac{\pi}{2}$ $\qquad 0 \leqq \sin\theta \leqq 1$

$f(\theta) = \sin 2\theta \cos\theta = 2\sin\theta\cos^2\theta = 2(\sin\theta - \sin^3\theta)$

$F(x) = x - x^3$ $\quad F'(x) = 1 - 3x^2 = (1-\sqrt{3}x)(1+\sqrt{3}x)$

最小値 $f(0) = 0$ 最大値 $f(\sin^{-1}\frac{1}{\sqrt{3}}) = \frac{4}{3\sqrt{3}}$

8.8



$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\frac{a^2}{9} + \frac{b^2}{4} = 1$ $\qquad b = \frac{2}{3}\sqrt{9-a^2}$

$V = 4ab = \frac{8}{3}a\sqrt{9-a^2}$

$V' = \frac{8}{3}\left(\sqrt{9-a^2} - \frac{a^2}{\sqrt{9-a^2}}\right) = \frac{8}{3}\frac{9-2a^2}{\sqrt{9-a^2}}$

$a = \frac{3}{\sqrt{2}}$ $\quad b = \sqrt{2}$ のとき 最大値

$\therefore \frac{b}{a} = \frac{\sqrt{2}}{\frac{3}{\sqrt{2}}} = \frac{2}{3}$

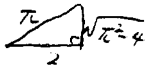
8.9 (1) $f(x) = \cos x - \frac{2}{\pi}$ $\qquad f(x)$ は $0 \leqq x \leqq \pi$ で減少

$f(0) = 1 - \frac{2}{\pi} > 0$ $\quad f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{2}{\pi} > 0$ $\quad f(\frac{\pi}{2}) = \frac{-2}{\pi} < 0$

$\therefore f(a) = 0$ のとき $\quad \frac{\pi}{4} < a < \frac{\pi}{2}$

(2) $F(x) = \sin x - \frac{2}{\pi}x$ $\qquad 0 \leqq x \leqq \frac{\pi}{2}$

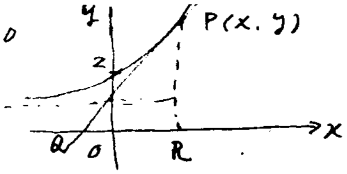
$F'(x) = \cos x - \frac{2}{\pi}$ $\qquad x = a = \cos^{-1}\frac{2}{\pi}$ のとき 最大値



$\sin a - \frac{2}{\pi}a$

(3) $M + \frac{2}{\pi}a = \sin a - \frac{2}{\pi}a + \frac{2}{\pi}a = \sin a = \sqrt{1 - \frac{4}{\pi^2}}$

8.10



$y = e^x + 1$

$PQ:\ Y = e^x(X - x) + e^x + 1$

$Q:\ e^x(X - x) = -e^x - 1 \quad X = -1 - \frac{1}{e^x} + x$

$\therefore\ Q(-1 + x - \frac{1}{e^x},\ 0) \quad R(x,\ 0)$

$S = \triangle PQR = \frac{1}{2}(1 + e^{-x})(1 + e^x) = \frac{1}{2}(2 + e^x + e^{-x})$

$S' = \frac{1}{2}(e^x - e^{-x}) \quad S'' = \frac{1}{2}(e^x + e^{-x})$

$\therefore\ x = 0,\ P(0, 2)$ のとき最小値 2

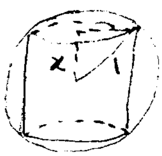
8.11 $x + 2y = 6$

$f(x) = \log x + \log y = \log x + \log(3 - \frac{x}{2})$

$f'(x) = \frac{1}{x} - \frac{2}{6 - x} \cdot \frac{1}{2} = \frac{6 - 2x}{x(6 - x)}$

$x = 3$ のとき最大値 $f(3) = \log 3 + \log \frac{3}{2} = \log \frac{9}{2}$

8.12



$r^2 = 1 - x^2$

$V = \pi r^2 \cdot 2x = 2\pi(x - x^3)$

$V' = 2\pi(1 - 3x^2)$

$x = \frac{1}{\sqrt{3}}$ のとき最大値 $\frac{4\pi}{3\sqrt{3}}$

8.13 $x^2 + y^4 = 1 \quad f(x, y) = x^3 y^4$

$y^4 = 1 - x^2 \quad F(x) = x^3(1 - x^2) = x^3 - x^5 \quad -1 \leqq x \leqq 1$

$F'(x) = 3x^2 - 5x^4 = 5x^2(\frac{3}{5} - x^2)$

$x = \sqrt{\frac{3}{5}}$ のとき最大値 $\frac{6\sqrt{3}}{25\sqrt{5}}$

$x = -\sqrt{\frac{3}{5}}$ のとき最小値 $-\frac{6\sqrt{3}}{25\sqrt{5}}$

8.14 $F(a) = \sum_{i=1}^{n} (x_i - a)^2 f_i$

$F'(a) = -2\sum_{i=1}^{n}(x_i - a) f_i = 0 \qquad a = \frac{\sum x_i f_i}{\sum f_i} = \bar{x}$ のとき最小

§9. 微分の応用(不等式)

9.1　$x>1,\ \alpha>1$ のとき

$f(x) = x^\alpha - 1 - \alpha(x-1)$

$f'(x) = \alpha x^{\alpha-1} - \alpha = \alpha(x^{\alpha-1}-1)$

$x>1$ のとき $f'(x)>0$ ∴ $f(x)$ 増加

∴ $f(x) > f(1) = 0$

∴ $\alpha(x-1) < x^\alpha - 1$ ・・・(1)

$g(x) = \alpha x^{\alpha-1}(x-1) - (x^\alpha - 1)$

$= \alpha x^\alpha - \alpha x^{\alpha-1} - x^\alpha + 1$

$= (\alpha-1)x^\alpha - \alpha x^{\alpha-1} + 1$

$g'(x) = \alpha(\alpha-1)x^{\alpha-1} - \alpha(\alpha-1)x^{\alpha-2}$

$= \alpha(\alpha-1)(x-1)x^{\alpha-2}$

$x>1$ のとき $g'(x)>0$ ∴ $g(x)$ は増加

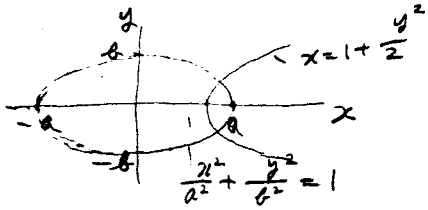
∴ $g(x) > g(1) = 0$

∴ $\alpha x^{\alpha-1}(x-1) > x^\alpha - 1$ ・・・(2)

(1), (2) より

$\alpha(x-1) < x^\alpha - 1 < \alpha x^{\alpha-1}(x-1)$

9.2　$x = 1 + \frac{y^2}{2}$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$a^2 \geqq 1$

交点 (x_1, y_1) とする

$\frac{dx}{dy} = y_1$　　$\frac{x_1}{a^2}\frac{dx}{dy} + \frac{y_1}{b^2} = 0$　　$\frac{dx}{dy} = -\frac{a^2 y_1}{b^2 x_1}$

直交するから $-\frac{a^2 y_1^2}{b^2 x_1} = -1$　∴ $b^2 x_1 = a^2 y_1^2$　$x_1 = 1 + \frac{y_1^2}{2}$

∴ $b^2\left(1+\frac{y_1^2}{2}\right) = a^2 y_1^2$　$2b^2 + b^2 y_1^2 = 2a^2 y_1^2$　$2b^2 = (2a^2 - b^2) y_1^2$

∴ $\underline{2a^2 - b^2 > 0}$

$b^2 x_1 = a^2 y_1^2$, $b^2 x_1^2 + a^2 y_1^2 = a^2 b^2$　∴ $b^2x_1^2 + b^2 x_1 - a^2 b^2 = 0$

$x_1^2 + x_1 - a^2 = 0$　∴ $x_1 = \frac{-1 \pm \sqrt{1+4a^2}}{2} > 1$

$\sqrt{1+4a^2} > 3$　$4a^2 > 8$　$\underline{a^2>2}$

∴ $2a^2 - b^2 > 0$, $a^2 > 2$

9.3　$f(x) = \cos x - \left(1 - \frac{x^2}{2}\right)$　$\left(0 < x \leqq \frac{\pi}{2}\right)$ とおく

$f'(x) = -\sin x + x$　$0<x\leqq\frac{\pi}{2}$ のとき $f'(x)>0$

$f(x)$ は増加　$x>0$ のとき

∴ $f(x) > f(0) = 0$

∴ $\cos x > 1 - \frac{x^2}{2}$

9.4 $f(x)=x-2\sin x-10$ とおくと

$f'(x)=1-2\cos x$ $\quad f'(x)=0$ $\quad \cos x=\frac{1}{2}$ $\quad x=2n\pi\pm\frac{\pi}{3}$

$f(4\pi-\frac{\pi}{3})=4\pi-\frac{\pi}{3}+\sqrt{3}-10>0$

$f(2\pi+\frac{\pi}{3})=2\pi+\frac{\pi}{3}-\sqrt{3}-10<0$

∴ $[2\pi+\frac{\pi}{3},\ 4\pi-\frac{\pi}{3}]$ に $f(x)=0$ の解がある。

9.5 $f(x)$ が 区間 $[a,b]$ で連続で $f(a)\neq f(b)$ ならば

$f(a)<f(b)$ のとき $\quad f(a)<{}^{\forall}m<f(b)$ に対して $\quad m=f(c)$ $\quad a<{}^{\exists}c<b$

$f(a)>f(b)$ のとき $\quad f(a)>{}^{\forall}m>f(b)$ 〃 $\quad m=f(c)$ $\quad a<{}^{\exists}c<b$

9.6 (1) $f(x)=x-\sin x$ $\quad 0<x<2\pi$

$f'(x)=1-\cos x\geqq 0$ ∴ $f(x)$ は増加

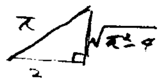
∴ $f(x)>f(0)=0$

∴ $x>\sin x$

(2) $g(x)=\sin x-\frac{2}{\pi}x$ $\quad 0<x<\frac{\pi}{2}$

$g'(x)=\cos x-\frac{2}{\pi}=0$

$x=\cos^{-1}\frac{2}{\pi}$ で $g(x)$ は最大となる



∴ $0<x<\frac{\pi}{2}$ のとき

$g(0)\leqq g(x)$, $\quad g(\frac{\pi}{2})\leqq g(x)$

$0\leqq g(x)$ $\quad 0\leqq g(x)$

$\frac{2}{\pi}x<\sin x$ $\quad$ (1)と合せて $\quad \frac{2}{\pi}x<\sin x<x$

9.7 $f(x)=\cos x$ とおくと $\quad f'(x)=-\sin x$ $\quad f''(x)=-\cos x$ $\quad f'''(x)=\sin x$

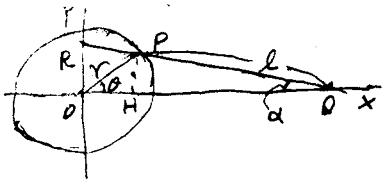
$f^{(4)}(x)=\cos x$

$\cos x=1-\frac{1}{2}x^2+\frac{x^4}{4!}-\frac{x^6}{6!}\cos\theta x$

$1-\frac{x^2}{2}<\cos x<1-\frac{x^2}{2}+\frac{x^4}{4!}$

§10 微分の応用(力学)

10.1



$\theta = \omega t$

$HP = r\sin\omega t \quad HP = l\sin\alpha$

$\therefore \sin\alpha = \dfrac{r}{l}\sin\omega t$ ---- ①

$OQ = r\cos\omega t + l\cos\alpha \quad OR = OQ\tan\alpha$

$v = \dfrac{d}{dt}OQ = -r\omega\sin\omega t - l\sin\alpha\dfrac{d\alpha}{dt}$ ①より $\cos\alpha\dfrac{d\alpha}{dt} = \dfrac{r}{l}\omega\cos\omega t$

$= -r\omega\sin\omega t - l\sin\alpha\dfrac{\omega r\cos\omega t}{l\cos\alpha}$

$= -r\omega(\sin\omega t + \tan\alpha\cos\omega t)$ $\quad \sin\omega t = \dfrac{l}{r}\sin\alpha$

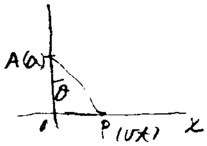
$= -\omega(l\sin\alpha + r\tan\alpha\cos\omega t)$ -- ②

$OR = OQ\tan\alpha = (r\cos\omega t + l\cos\alpha)\tan\alpha$ ③

$= (r\tan\alpha\cos\omega t + l\sin\alpha)$ -

②, ③より $v = -\omega\cdot OR$

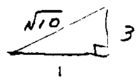
10.2



$P(3a, 0)$

$\tan\theta = \dfrac{vt}{a}$ $\quad vt = 3a \quad t = \dfrac{3}{v}a$

$\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{v}{a}$ $\quad \tan\theta = 3$



$\therefore \dfrac{d\theta}{dt} = \dfrac{v}{a}\cos^2\theta = \dfrac{v}{a}\dfrac{1}{10}$

$= \dfrac{1}{10}\dfrac{v}{a}$

§11 n次導関数

11.1 (1) $f(x) = x^2 e^x$

$f^{(10)}(x) = x^2 e^x + 20x e^x + 90 e^x$

(2) $f(x) = e^x \sin x$

$f^{(8)}(x) = e^x \sin x + 8e^x \cos x - 28 e^x \sin x - 56 e^x \cos x + 70 e^x \sin x$
$+ 56 e^x \cos x - 28 e^x \sin x - 8 e^x \cos x + e^x \sin x$
$= 16 e^x \sin x$

11.2 (1) $y = x^2 \cos x$

$y^{(n)} = x^2 \cos(x + \frac{n}{2}\pi) + 2nx \cos(x + \frac{n-1}{2}\pi) + n(n-1)\cos(x + \frac{n-2}{2}\pi)$

(2) $y = e^x \sin x$

$y' = (\sin x + \cos x) e^x = \sqrt{2}\, e^x \sin(x + \frac{\pi}{4})$

$y^{(n)} = (\sqrt{2})^n e^x \sin(x + \frac{n}{4}\pi)$

(3) $y = x^{n-1} \log x$

$$y^{(n)} = \sum_{k=0}^{n-1} {}_nC_k\, (x^{n-1})^{(k)} (\log x)^{(n-k)}$$

$$= \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} (n-1)(n-2)\cdots(n-k)\, x^{n-k-1} \cdot (-1)^{n-k-1} (n-k-1)! \frac{1}{x^{n-k}}$$

$$= \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{n!(n-1)!}{k!(n-k)!} \frac{1}{x} = \frac{-(n-1)!}{x} \sum_{k=1}^{n-1} \frac{(-1)^{n-k} n!}{k!(n-k)!}$$

$$= -\frac{(n-1)!}{x} \sum_{k=0}^{n-1} {}_nC_k (-1)^{n-k}$$

$(1-1)^n = \sum_{k=0}^{n} {}_nC_k (-1)^{n-k}$ $\quad \therefore \sum_{k=0}^{n-1} {}_nC_k (-1)^{n-k} = -{}_nC_n = -1$

$$y^{(n)} = \frac{(n-1)!}{x}$$

(4) $y = x^3 \sin x$

$y^{(n)} = x^3 \sin(x + \frac{n}{2}\pi) + 3nx^2 \sin(x + \frac{n-1}{2}\pi) + 3n(n-1)x \sin(x + \frac{n-2}{2}\pi)$
$+ n(n-1)(n-2) \sin(x + \frac{n-3}{2}\pi)$

(5) $y = \frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$ $\quad y' = \frac{1}{2}(1-x)^{-\frac{3}{2}}$ $\quad y'' = \frac{1}{2}\cdot\frac{3}{2}(1-x)^{\frac{5}{2}}$

$$y^{(n)} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n} (1-x)^{-\frac{2n-1}{2}}$$

(6) $y = x \sin x$

$y^{(n)} = x \sin(x + \frac{n}{2}\pi) + n \sin(x + \frac{n-1}{2}\pi)$

11.3 (1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

(2) $\sqrt[3]{1+x^2} = 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4$

(3) $\log(1+x+x^2) = x + x^2 - \frac{1}{2}(x+x^2)^2 + \frac{1}{3}(x+x^2)^3$

$= x + \frac{1}{2}x^2 - \frac{2}{3}x^3$

(4) $\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^3$

11.4 (1) $f(x) = \tan^{-1}x \qquad f'(x) = \frac{1}{1+x^2}$

$(1+x^2)f'(x) = 1 \qquad (1+x^2)f^{(n)}(x) + 2(n-1)xf^{(n-1)}(x) + (n-1)(n-2)f^{(n-2)}(x) = 0$

$\therefore f^{(n)}(0) = -(n-1)(n-2)f^{(n-2)}(0)$

$f(0) = 0 \qquad f'(0) = 1$

$f^{(2n)}(0) = 0 \qquad f^{(2n+1)}(0) = (-1)^n(2n)!$

11.5 $f(x) = -x^3\cos x$

$= -x^3\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!}\right)$

$= -x^3 + \frac{x^5}{2!} - \frac{x^7}{4!} + \cdots + (-1)^{n+1}\frac{x^{2n+3}}{(2n)!}$

11.6 平均値の定理より

$f(x) - f(a) = (x-a)f'\{a+\theta(x-a)\} \qquad 0<\theta<1$

$f'(x) = 0$ ならば

$f(x) = f(a)$ 定数

11.7 $(x^{n-1}e^{\frac{1}{x}})^{(n)} = (-1)^n e^{\frac{1}{x}}/x^{n+1}$

数学的帰納法

[I] $n=1$ のとき

$(e^{\frac{1}{x}})' = -\frac{e^{\frac{1}{x}}}{x^2} = (-1)^1 e^{\frac{1}{x}}/x^{1+1}$

成り立つ

$n=2$ のとき

$(xe^{\frac{1}{x}})'' = (e^{\frac{1}{x}} - \frac{1}{x}e^{\frac{1}{x}})' = \frac{1}{x^3}e^{\frac{1}{x}} = (-1)^2 e^{\frac{1}{x}}/x^{2+1}$

$\therefore$ 成り立つ

[II] $n=k-1,\ k$ のとき成り立つとする

$(x^{k-2}e^{\frac{1}{x}})^{(k-1)} = (-1)^{k-1}e^{\frac{1}{x}}/x^k$

$(x^{k-1}e^{\frac{1}{x}})^{(k)} = (-1)^k e^{\frac{1}{x}}/x^{k+1}$

$n=k+1$ のとき

$$(x^k e^{\frac{1}{x}})^{(k+1)} = (k x^{k-1} e^{\frac{1}{x}} - x^{k-2} e^{\frac{1}{x}})^{(k)}$$
$$= k(x^{k-1} e^{\frac{1}{x}})^{(k)} - (x^{k-2} e^{\frac{1}{x}})^{(k)}$$
$$= (-1)^k k\, e^{\frac{1}{x}}/x^{k+1} - \{(-1)^{k-1} e^{\frac{1}{x}}/x^k\}'$$
$$= (-1)^k k\, e^{\frac{1}{x}}/x^{k+1} - (-1)^k e^{\frac{1}{x}}/x^{k+2} - (-1)^k k\, e^{\frac{1}{x}}/x^{k+1}$$
$$= (-1)^{k+1} e^{\frac{1}{x}}/x^{k+2}$$

∴ 成り立つ

[I].[II] より $(x^{n-1} e^{\frac{1}{x}})^{(n)} = (-1)^n e^{\frac{1}{x}}/x^{n+1}$

11.8 $y = \cos^{-1} x$

(1) $y' = \dfrac{-1}{\sqrt{1-x^2}}$

(2) $y'' = \dfrac{-x}{(\sqrt{1-x^2})^3}$ ∴ $(1-x^2) y'' = -x \dfrac{1}{\sqrt{1-x^2}} = x y'$

∴ $(1-x^2) y'' - x y' = 0$

(3) $(1-x^2) y^{(n+2)} - 2nx\, y^{(n+1)} - n(n-1) y^{(n)} - x y^{(n+1)} - n y^{(n)} = 0$

$(1-x^2) y^{(n+2)} - (2n+1) x\, y^{(n+1)} - n^2 y^{(n)} = 0$

(4) $x=0$ を代入

$y^{(n+2)}(0) = n^2 y^{(n)}(0)$ $y(0) = \dfrac{\pi}{2}$ $y'(0) = -1$ $y''(0) = 0$

∴ $y^{(2m)}(0) = (2m-2)^2 (2m-4)^2 \cdots 2^2 \cdot y''(0) = 0$

$y^{(2m+1)}(0) = (2m-1)^2 (2m-3)^2 \cdots 3^2 \cdot 1^2 (-1)$

11.9 $y = \tan^{-1} x$

(1) $y' = \dfrac{1}{1+x^2}$ $(1+x^2) y' = 1$ $(1+x^2) y'' + 2x y' = 0$

(2) $(1+x^2) y^{(n+2)} + 2nx\, y^{(n+1)} + n(n-1) y^{(n)} + 2x y^{(n+1)} + 2n y^{(n)} = 0$

$(1+x^2) y^{(n+2)} + 2(n+1) x\, y^{(n+1)} + n(n+1) y^{(n)} = 0$

(3) $y^{(n+2)}(0) + n(n+1) y^{(n)}(0) = 0$ $y(0) = 0$ $y'(0) = 1$ $y''(0) = 0$

$y^{(n+2)} = -n(n+1) y^{(n)}(0)$

$y^{(2m)}(0) = 0$

$y^{(2m+1)} = (-1)^m (2m)!\, y'(0) = (-1)^m (2m)!$

11.10 $f(x)=e^x \quad f^{(n)}(x)=e^x \quad f^{(n)}(0)=1$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+ \quad +\frac{x^n}{n!}+\frac{e^{\theta x}x^{n+1}}{(n+1)!} \qquad 0<\theta<1$$

$\therefore$ $x>0$ のとき $\dfrac{e^{\theta x}x^{n+1}}{(n+1)!}>0$

$$e^x>1+x+\frac{x^2}{2!}+ \quad +\frac{x^n}{n!}$$

11.11. (1) $y=\sin x=x-\dfrac{x^3}{3!}$

(2) $\displaystyle\lim_{x\to 0}\left\{\frac{\sin x-(a+bx+cx^2)}{x^3}\right\}=d \quad (d\neq 0)$

$\displaystyle\lim_{x\to 0}\{\sin x-(a+bx+cx^2)\}=-a=0 \qquad \therefore a=0$

$\displaystyle\lim_{x\to 0}\{\cos x-(b+2cx)\}=1-b=0 \qquad \therefore b=1$

$\displaystyle\lim_{x\to 0}(-\sin x-2c)=-2c=0 \qquad \therefore c=0$

$\displaystyle\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}=d \qquad d=-\frac{1}{6}$

11.12 $F(x)=f(x)-\left\{f(a)+f'(a)(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+ \cdots +\dfrac{f^{(n)}(a)}{n!}(x-a)^n\right\}$

$G(x)=(x-a)^{n+1}$ とおくと

$F(a)=F'(a)=F''(a)= \cdots =F^{(n)}(a)=0$

$G(a)=G'(a)=G''(a)= \quad =G^{(n)}(a)=0$

$\therefore$ コーシーの平均値の定理より

$$\frac{F(x)}{G(x)}=\frac{F^{(n+1)}\{a+\theta(x-a)\}}{G^{(n+1)}\{a+\theta(x-a)\}}=\frac{f^{(n+1)}\{a+\theta(x-a)\}}{(n+1)!} \qquad 0<\theta<1$$

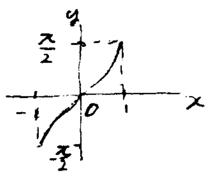
$\therefore F(x)=\dfrac{1}{(n+1)!}f^{(n+1)}\{a+\theta(x-a)\}(x-a)^{n+1}$

$\therefore f(x)=f(a)+f'(a)(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+ \cdots +\dfrac{f^{(n)}(a)}{n!}(x-a)^n+R_n$

$$R_n=\frac{1}{(n+1)!}f^{(n+1)}\{a+\theta(x-a)\}(x-a)^{n+1}$$

11.13 9.7と同じ

11.14 (1) $y=\sin^{-1}x$



(2) $y'=\dfrac{1}{\sqrt{1-x^2}}$

$y''=\dfrac{x}{(\sqrt{1-x^2})^3} \qquad (1-x^2)y''=xy'$

(3) $(1-x^2)y^{(n+2)}(x)-(2n+1)xy^{(n+1)}(x)-n^2y^{(n)}(x)=0$

(2)より $(1-x^2)y^{(3)}-3xy''(x)-y'(x)=0$

[I] $n=1$ のとき 成り立つ

[II] $n=k$ のとき 成り立つとする

$(1-x^2)y^{(k+2)}(x)-(2k+1)xy^{(k+1)}(x)-k^2y^{(k)}(x)=0$

両辺を x で微分して

$(1-x^2)y^{(k+3)}(x)-2xy^{(k+2)}(x)-(2k+1)xy^{(k+2)}(x)-(2k+1)y^{(k+1)}(x)-k^2y^{(k+1)}(x)=0$

$(1-x^2)y^{(k+3)}(x)-(2k+3)xy^{(k+2)}(x)-(k+1)^2y^{(k+1)}(x)=0$

∴ $n=k+1$ のとき 成り立つ

[I],[II]より すべての $n\geqq 1$ に対して 成り立つ

11.15 $f_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\{(x^2-1)^n\}$

(1) $(x^2-1)^n=\sum_{k=0}^{n}(-1)^{n-k}{}_nC_k\,x^{2k}$ $\quad x^{2k}$ の係数 $(-1)^{n-k}{}_nC_k$

(2) $\dfrac{d^n}{dx^n}\{(x^2-1)^n\}=\sum_{k=[\frac{n+1}{2}]}^{n}(-1)^{n-k}\dfrac{n!}{k!(n-k)!}2k(2k-1)\cdots(2k-n+1)x^{2k-n}$

$f_n(x)=\dfrac{1}{2^n n!}\sum_{k=[\frac{n+1}{2}]}^{n}(-1)^{n-k}\dfrac{n!}{k!(n-k)!}2k(2k-1)\cdots(2k-n+1)x^{2k-n}$

$n=2m+1$ のとき $f_{2m+1}(0)=0$

$n=2m$ のとき $f_{2m}(0)=\dfrac{1}{2^{2m}(2m)!}(-1)^m\dfrac{(2m)!}{m!\,m!}(2m)!$

$=\dfrac{(-1)^m(2m)!}{2^{2m}(m!)^2}$

(3) $n=2m$ のとき

$\dfrac{f_{2m+2}(0)}{f_{2m}(0)}=\dfrac{(-1)^{m+1}(2m+2)!}{2^{2m}\cdot 2^2\{(m+1)!\}^2}\dfrac{2^{2m}(m!)^2}{(-1)^m(2m)!}$

$=-\dfrac{(2m+2)(2m+1)}{2^2(m+1)^2}=-\dfrac{2m+1}{2(m+1)}=-\dfrac{2m+1}{2m+2}$

11.16 $F_n(x) = e^{-x^2+x}\left(e^{x^2-x}\right)^{(n)}$

(1) $\left(e^{x^2-x}\right)' = (2x-1)e^{x^2-x}$ 　　$g(x) = e^{x^2-x}$ とおくと

$g'(x) = (2x-1)g(x)$

$\therefore g^{(n+1)}(x) = (2x-1)g^{(n)}(x) + 2n\cdot g^{(n-1)}(x)$

$\therefore e^{-x^2+x}g^{(n+1)}(x) = (2x-1)e^{-x^2+x}g^{(n)}(x) + 2n\, e^{-x^2+x}g^{(n-1)}(x)$

$\therefore F_{n+1}(x) = (2x-1)F_n(x) + 2n F_{n-1}(x)$

(2) $F_n'(x) = (-2x+1)e^{-x^2+x}\left(e^{x^2-x}\right)^{(n)} + e^{-x^2+x}\left(e^{x^2-x}\right)^{(n+1)}$

$= (-2x+1)F_n(x) + F_{n+1}(x)$

$= (-2x+1)F_n(x) + (2x-1)F_n(x) + 2nF_{n-1}(x)$ 　((1) より)

$= 2nF_{n-1}(x)$

$\therefore F_n'(x) = 2nF_{n-1}(x)$

(3) (1), (2) より

$F_{n+1}(x) = (2x-1)F_n(x) + 2nF_{n-1}(x)$

$= (2x-1)F_n(x) + F_n'(x)$

$\therefore F_{n+1}'(x) = 2F_n(x) + (2x-1)F_n'(x) + F_n''(x)$

(2) より

$F_{n+1}'(x) = 2(n+1)F_n(x)$

$\therefore (2n+2)F_n(x) = 2F_n(x) + (2x-1)F_n'(x) + F_n''(x)$

$\therefore F_n''(x) + (2x-1)F_n'(x) - 2nF_n(x) = 0$

§12 近似式

12.1 $x \fallingdotseq 0$ のとき

(1) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$

$\therefore \log(1+x) \fallingdotseq x - \frac{x^2}{2} + \frac{x^3}{3}$

(2) $\sqrt{1-x+x^2} = \{1-(x-x^2)\}^{\frac{1}{2}}$

$= 1 - \frac{1}{2}(x-x^2) - \frac{1}{8}(x-x^2)^2 + O(x^3)$

$= 1 - \frac{x}{2} + \frac{3}{8}x^2 + O(x^3)$

(3) $f(x) = e^x \cos x$ とおくと

$f'(x) = e^x(\cos x - \sin x) = \sqrt{2}\, e^x \cos(x + \frac{\pi}{4})$

$f^{(n)}(x) = (\sqrt{2})^n e^x \cos(x + \frac{n}{4}\pi)$

$f(0) = 1 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = -2 \quad f^{(4)}(0) = -4$

$e^x \cos x = 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + O(x^5)$

12.2 (1) $\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$

$\fallingdotseq 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$

(2) $f(x) = 4^x$ とおくと

$f^{(n)}(x) = 4^x (\log 4)^n$

$4^x \fallingdotseq 1 + x \log 4 + \frac{(\log 4)^2}{2}x^2 + \frac{(\log 4)^3}{6}x^3$

12.3 $-1 < x < 1$

$(1+x)^m \fallingdotseq 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{n}x^n$

12.4 $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}\cos(\theta x)$

誤差 $E = -\frac{x^6}{6!}\cos(\theta x)$

$\therefore |E| = \frac{x^6}{6!}|\cos \theta x| \leq \frac{x^6}{6!}$

12.5 $(e^{\sin x} - 1)/\sin x$

$e^{\sin x} = 1 + \sin x + \frac{1}{2!}\sin^2 x + \frac{1}{3!}\sin^3 x + \cdots$

$\frac{e^{\sin x} - 1}{\sin x} = 1 + \frac{1}{2!}\sin x + O(\sin^2 x)$

$\therefore \lim_{x \to 0} \frac{e^{\sin x} - 1}{\sin x} = 1$

12.6 $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\sin\left(\theta x + \frac{5}{2}x\right)$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}\cos(\theta x)$$

$$\therefore \left|\sin x - \left(x - \frac{x^3}{3!}\right)\right| = \left|\frac{x^5}{5!}\right| |\cos\theta x| \leq \frac{|x|^5}{5!}$$

12.7 $\quad 0 \leqq x \leqq \pi$

(1) $\quad f(x) = 1 - \frac{x^2}{2} \quad g(x) = \cos x \quad h(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

$F(x) = g(x) - f(x)$ とおく

$F'(x) = -\sin x + x > 0 \quad \therefore F(x)$ は増加

$0 \leqq x \leqq \pi$ のとき $\quad F(x) \geqq F(0) = 0 \quad \therefore F(x) \geqq 0$

$\therefore f(x) \leqq g(x)$

$G(x) = h(x) - g(x)$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\cos\theta x\right)$$

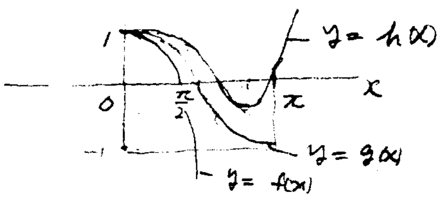
$$= \frac{x^4}{24}(1 - \cos\theta x) \geqq 0$$

$\therefore h(x) - g(x) \geqq 0$

$h(x) \geqq g(x)$

$\therefore f(x) \leqq g(x) \leqq h(x)$

(2)



(3) $\quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$

$x = \frac{2}{10} \qquad \frac{1}{24}\,\frac{2^4}{10^4} = \frac{2}{3\cdot 10^4} < 0.7 \times \frac{1}{10^4} = 0.00\ldots$

§13 テイラー展開

13.1 $f(x)=\sin x \qquad f^{(n)}(x)=\sin(x+\frac{n}{2}\pi)$

$f^{(2n+1)}(0)=(-1)^n \qquad f^{(2n)}(0)=0$

$$\lim_{n\to\infty}\frac{x^{2n+1}}{(2n+1)!}\sin(\theta x)=0$$

$$\therefore \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\cdots$$

13.2 (1) $\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}$

$$=1+(-\tfrac{1}{2})(-x^2)+\tfrac{1}{2!}(-\tfrac{1}{2})(-\tfrac{3}{2})(-x^2)^2+\tfrac{1}{3!}(-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})(-x^2)^3+\cdots$$

$$=1+\frac{x^2}{2}+\frac{1\cdot 3}{2!}\left(\frac{x^2}{2}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{x^2}{2}\right)^3+\cdots+\frac{1\cdot 3\cdots(2n-1)}{n!}\left(\frac{x^2}{2}\right)^n+\cdots \qquad |x|<1$$

(2) $\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x$

$$\sin^{-1}x=x+\frac{x^3}{3\cdot 2}+\frac{1\cdot 3\,x^5}{5\cdot 2!\,2^2}+\frac{1\cdot 3\cdot 5\,x^7}{7\,3!\,2^3}+\cdots+\frac{1\cdot 3\cdots(2n-1)\,x^{2n+1}}{(2n+1)\,n!\,2^n}+\cdots \qquad |x|<1$$

(3) $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}-\cdots+(-1)^n\frac{x^n}{n}+\cdots \qquad |x|<1$

(4) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\cdots$

13.3 (1) $\frac{x}{1+x^2}=x(1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}-\cdots) \qquad |x|<1$

$$=x-x^3+x^5-x^7+\cdots+(-1)^n x^{2n+1}+\cdots$$

(2) $\{\log(x+\sqrt{1+x^2})\}'=\frac{1}{\sqrt{1+x^2}}=(1+x^2)^{-\frac{1}{2}}$

$$\frac{1}{\sqrt{1+x^2}}=1-\frac{1}{2}x^2+\frac{1\cdot 3\,x^4}{2!\,2^2}+\frac{(-1)^3\,1\cdot 3\cdot 5\,x^6}{3!\,2^3}+\cdots+\frac{(-1)^n\,1\cdot 3\cdots(2n-1)}{n!\,2^n}x^{2n}+\cdots$$

$$\log(x+\sqrt{1+x^2})=1-\frac{x^3}{3\cdot 2}+\frac{1\cdot 3\,x^5}{5\cdot 3!\,2^2}+\cdots+\frac{(-1)^n\,1\cdot 3\cdots(2n-1)}{(2n+1)\,n!\,2^n}x^{2n+1}+\cdots \qquad |x|<1$$

(3) $\{\log(1-x)\}'=\frac{-1}{1-x}=-(1+x+x^2+x^3+\cdots+x^n+\cdots)$

$$\log(1-x)=-(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{1}{n+1}x^{n+1}+\cdots) \qquad |x|<1$$

(4) $\left(\frac{1}{1-x}\right)'=\frac{1}{(1-x)^2}$

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots \qquad |x|<1$$

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\cdots+nx^{n-1}+\cdots$$

(5) $\left(\frac{1}{(1-x)^2}\right)'=\frac{2}{(1-x)^3}$

$$\frac{1}{(1-x)^3}=\frac{1}{2}(2+3\cdot 2x+4\cdot 3x^2+\cdots+n(n-1)x^{n-2}+\cdots) \qquad |x|<1$$

(6) $(\log(1+x))' = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n}{n+1} x^{n+1} + \cdots$

(7) $\sin 3x + x \cos 2x$

$= 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \cdots + (-1)^{n-1} \frac{(3x)^{2n-1}}{(2n-1)!} + \cdots$

$+ x\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots\right)$

$= 4x + (-1)\left(\frac{3^3}{3!} + \frac{2^2}{2!}\right) x^3 + (-1)^2 \left(\frac{3^5}{5!} + \frac{2^4}{4!}\right) x^5 + \cdots + (-1)^{n-1} \left(\frac{3^{2n-1}}{(2n-1)!} + \frac{2^{2n-2}}{(2n-2)!}\right) x^{2n-1} + \cdots$

13.4 $f(x) = \frac{x}{x^2+1}$ $\quad (x^2+1) f(x) = x \quad (x^2+1) f'(x) + 2x f(x) = 1$

$(x^2+1) f^{(n+1)}(x) + 2nx f^{(n)}(x) + n(n-1) f^{(n-1)}(x) + 2x f^{(n)}(x) + 2n f^{(n-1)}(x) = 0$

$(x^2+1) f^{(n+1)}(x) + 2(n+1) x f^{(n)}(x) + n(n+1) f^{(n-1)}(x) = 0$

$x = 0$ を代入して

$f^{(n+1)}(0) = -(n+1) n f^{(n-1)}(0) \qquad f(0) = 0 \quad f'(0) = 1$

$f^{(2m)}(0) = 0 \qquad f^{(2m+1)}(0) = (-1)^m (2m+1)!$

$\therefore \frac{x}{1+x^2} = x - x^3 + x^5 - \cdots + (-1)^m x^{2m+1} + \cdots$

13.5 $f(x) = \log(1+x) \qquad f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \qquad f(0) = 0 \quad f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$

13.6 $f(x) = \tan^{-1} x \qquad f'(x) = \frac{1}{1+x^2} \qquad (1+x^2) f'(x) = 1$

$(1+x^2) f^{(n+1)}(x) + 2nx f^{(n)}(x) + n(n-1) f^{(n-1)}(x) = 0$

$\therefore f^{(n+1)}(0) = -n(n-1) f^{(n-1)}(0) \qquad f(0) = 0 \quad f'(0) = 1$

$\therefore f^{(2m)}(0) = 0 \qquad f^{(2m+1)}(x) = (-1)^m (2m)!$

$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} - \cdots \qquad |x| < 1$

$x = 1$ を代入して

$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1} \frac{1}{2n-1} \cdots\right)$

13.7 $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots \qquad -1 < x \leq 1$

(1) $-\log(t-1) + 2\log t - \log(t+1)$

$= \log \frac{t^2}{t^2-1} = -\log \frac{t^2-1}{t^2} = -\log\left(1 - \frac{1}{t^2}\right) \qquad \frac{1}{t^2} < 1 \quad t^2 > 1$

$= -\left(-\frac{1}{t^2} - \frac{1}{2t^4} - \frac{1}{3t^6} - \cdots - \frac{1}{nt^{2n}} \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{nt^{2n}}$

(2) $2\log 3 - 3\log 2 = \log\frac{9}{8}$ (1) で $x=3$ を代入して

$$2\log 3 - 3\log 2 = \sum_{n=1}^{\infty}\frac{1}{n\cdot 3^{2n}}$$

$$= \frac{1}{9} + \frac{1}{2\cdot 81} + \frac{1}{3\cdot 729} + \cdots$$

$$\fallingdotseq 0.117$$

13.8 $f(x) = e^{\sqrt{3}x}\sin x$ $\quad f'(x) = (\sqrt{3}\sin x + \cos x)e^{\sqrt{3}x}$

$$= 2e^{\sqrt{3}x}\sin\left(x+\frac{\pi}{6}\right)$$

$\therefore$ $f^{(n)}(x) = 2^n e^{\sqrt{3}x}\sin\left(x+\frac{n}{6}\pi\right)$

$\therefore$ $e^{\sqrt{3}x}\sin x = \sum_{n=0}^{\infty}\frac{2^n}{n!}\left(\sin\frac{n}{6}\pi\right)x^n$

13.9 $\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{1}{2}(-x)^2 + \cdots$

$$= 1 - \frac{x}{2} - \frac{x^2}{8} \cdots$$

13.10 $f(x) = \log(1+x)$ とおくと

$f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ $\quad f^{(5)}(0) = 4!$ $\quad x^5 \cdot$ $\quad \frac{1}{5}$

$\quad f^{(6)}(0) = -5!$ $\quad x^6$ $\quad -\frac{1}{6}$

13.11 (1) $f(x) = \dfrac{1}{\sqrt{1+x^2}}\cdot\log(x+\sqrt{x^2+1})$

$f'(x) = \dfrac{-x}{(\sqrt{1+x^2})^3}\log(x+\sqrt{1+x^2}) + \dfrac{1}{(\sqrt{1+x^2})^2}$

$\therefore$ $(1+x^2)f'(x) = -\dfrac{x}{\sqrt{1+x^2}}\log(x+\sqrt{1+x^2}) + 1$

$(1+x^2)f'(x) + xf(x) = 1$

(2) $(n+1)$ 回微分して

$(1+x^2)f^{(n+2)}(x) + 2x(n+1)f^{(n+1)}(x) + (n+1)nf^{(n)}(x) + xf^{(n+1)}(x) + (n+1)f^{(n)}(x) = 0$

$(1+x^2)f^{(n+2)}(x) + (2n+3)xf^{(n+1)}(x) + (n+1)^2f^{(n)}(x) = 0$

(3) $f(0) = 0$ $\quad f'(0) = 1$

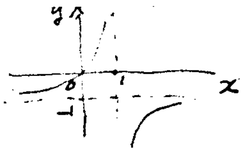
$x=0$ を代入して $\quad f^{(n+2)}(0) = -(n+1)^2f^{(n)}(0)$

$\therefore$ $f^{2m}(0) = 0$ $\quad f^{(2m+1)}(0) = -(2m)^2f^{(2m-1)}(0) = (-1)^m(2m)^2(2m-2)^2\cdots 2^2f'(0)$

$$= (-1)^m 2^{2m}(m!)^2$$

$$f(x) = \sum_{m=0}^{\infty}\frac{(-1)^m 2^{2m}(m!)^2}{(2m+1)!}x^{2m+1}$$

13.12 (1) $f(x) = \dfrac{x}{1-x} = -1 + \dfrac{1}{1-x}$



(2) $f(x) = x + x^2 + x^3 + \cdots + x^n + \cdots$

$|x| < 1, \ (-1 < x < 1)$

(3) $f'(x) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$

$-1 < x < 1$

13.13 $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots + \dfrac{f^{(n)}(0)}{n!}x^n + \cdots$

13.14 (1) $f(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

(2) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$

(3) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + \dfrac{(-1)^n x^{2n}}{(2n)!} + \cdots$

13.15 $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \cdots$

$\therefore \dfrac{d}{dx}$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$

13.16 $\sin k\theta = \dfrac{\sin(x\theta)\sin(y\theta)}{\sin\frac{\theta}{2}}$

$\sin k\theta \sin\frac{\theta}{2} = \sin(x\theta)\sin(y\theta)$

$\left(k\theta - \dfrac{k^3\theta^3}{3!} + \dfrac{k^5\theta^5}{5!} - \cdots\right)\left(\dfrac{\theta}{2} - \dfrac{1}{3!}\left(\dfrac{\theta}{2}\right)^3 + \cdots\right) = \left(x\theta - \dfrac{x^3\theta^3}{3!} + \cdots\right)\left(y\theta - \dfrac{y^3\theta^3}{3!} + \cdots\right)$

より $\dfrac{k}{2} = xy \qquad \dfrac{k}{8} + \dfrac{k^3}{2} = xy(x^2+y^2)$

$\therefore \dfrac{1}{4} + k^2 = x^2 + y^2$

$\begin{cases} x^2 + y^2 = k^2 + \frac{1}{4} \\ x^2y^2 = \frac{k^2}{4} \end{cases}$

$x = \pm\frac{1}{2} \ \ y = \pm k \ ; \quad x = \pm k, \ y = \pm\frac{1}{2}$

13.17 (1) $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$

(2) $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots$

$\therefore \tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots + (-1)^n \dfrac{x^{2n+1}}{2n+1} + \cdots$

3.18 (1) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$

(2) $f(x) = \sec x = \frac{1}{\cos x}$

$f'(x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$

$f''(x) = \sec x \tan^2 x + \sec^3 x = \sec x(\tan^2 x + \sec^2 x)$

$f'''(x) = \sec x\ (\tan^3 x + 5\sec^2 x \tan x)$

$f^{(4)}(x) = \sec x(\tan^4 x + 18\tan^2 x \sec^2 x + 5\sec^4 x)$

$f^{(5)}(x) = \sec x(\tan^5 x + 58\tan^3 x \sec^2 x + 61\sec^4 x \tan x)$

$\therefore f(0) = 1 \quad f'(0) = 0 \quad f''(0) = 1 \quad f^{(3)}(0) = 0 \quad f^{(4)}(0) = 5 \quad f^{(5)}(0) = 0$

$\therefore \sec x = 1 + \frac{1}{2}x^2 + \frac{5}{4!}x^4 + \cdots$

(3) $\frac{\sin x - x}{x^2} = -\frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots + \frac{(-1)^n x^{2n-1}}{(2n+1)!} + \cdots$

(4) $f(x) = \tan^{-1} x$

$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}$

$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)} + \cdots$

§14 総合問題

14.1 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(1) 接線 $y=mx+c$ ．だ円 $b^2x^2+a^2y^2=a^2b^2$

$\therefore\ b^2x^2+a^2(mx+c)^2-a^2b^2=0$

$(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$

$\frac{D}{4}=0 \quad a^4m^2c^2-a^2(c^2-b^2)(b^2+a^2m^2)=0$

$(a^2m^2-b^2-a^2m^2)c^2=-b^2(b^2+a^2m^2)$

$c^2=b^2+a^2m^2 \qquad c=\pm\sqrt{b^2+a^2m^2}$

(2) 直交する 2 接線を

$\left.\begin{array}{l} y=m_1x+c_1 \\ y=m_2x+c_2 \end{array}\right\}$ とすると $\qquad c_1=\pm\sqrt{m_1^2a^2+b^2}$, $c_2=\pm\sqrt{m_2^2a^2+b^2}$

$m_1\cdot m_2=-1 \quad \therefore\ m_2=-\frac{1}{m_1}$

$\therefore\ (m_1-m_2)x+c_1-c_2=0 \quad \therefore\ x=\frac{c_2-c_1}{m_1-m_2}=\frac{m_1(c_2-c_1)}{m_1^2+1}$

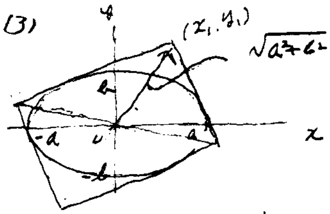
$y=\frac{m_1^2c_2+c_1}{m_1^2+1}$

$x^2+y^2=\frac{m_1^2c_2^2+c_1^2}{m_1^2+1}$

$m_1^2c_2^2+c_1^2=a^2+m_1^2b^2+m_1^2a^2+b^2=(m_1^2+1)(a^2+b^2)$

$\therefore\ x^2+y^2=a^2+b^2$

(3)



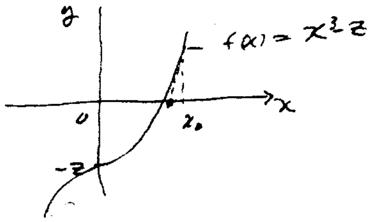
(2)より $x^2+y^2=a^2+b^2$

$2x_1\cdot 2y_1 \leqq \frac{1}{2}\{(2x_1)^2+(2y_1)^2\}$

$\leqq 2(a^2+b^2)$

$\therefore$ 最大値 $2(a^2+b^2)$

*4. 2. $\sqrt[3]{z}$



$x_n = x_{n-1} - F(x_{n-1})$

$f(x) = x^3 - z$ $\quad f(x_0) = x_0^3 - z > 0$

$f'(x) = 3x^2$

$y - f(x_0) = f'(x_0)(x - x_0)$ $\quad x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - z}{3x_0^2} = \frac{2}{3}x_0 + \frac{z}{3x_0^2}$

$x_n = x_{n-1} - \frac{x_{n-1}^3 - z}{3x_{n-1}^2} = \frac{2}{3}x_{n-1} + \frac{z}{3x_{n-1}^2}$

$x_n - x_{n-1} = -\frac{f(x_{n-1})}{f'(x_{n-1})} < 0$ $\quad f'(x_{n-1}) > 0,\ f(x_{n-1}) > 0.$

$\therefore\ x_{n-1} > x_n$

∴ 数列 $\{x_n\}$ は単調減少

$\therefore\ \lim_{n\to\infty} x_n = \alpha$ とおくと $\quad x_n = \frac{2}{3}x_{n-1} + \frac{z}{3x_{n-1}^2}$ より

$\alpha = \frac{2}{3}\alpha + \frac{z}{3\alpha^2}$ $\quad 3\alpha^3 = 2\alpha^3 + z$ $\quad \therefore\ \alpha^3 - z = 0.$

14.3 $\quad y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$ $\quad \therefore\ y(e^{2x}+1) = e^{2x} - 1$

$(y-1)e^{2x} = -1 - y$ $\quad e^{2x} = \frac{1+y}{1-y}$ $\quad -1 < y < 1$

$2x = \log\frac{1+y}{1-y}$ $\quad x = \frac{1}{2}\log\frac{1+y}{1-y}$

$\therefore\ y = \frac{1}{2}\log\frac{1+x}{1-x}$

$y' = \frac{1}{2}\left\{\frac{1}{1+x} + \frac{1}{1-x}\right\} = \frac{1}{1-x^2}$