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## 第7章 重積分

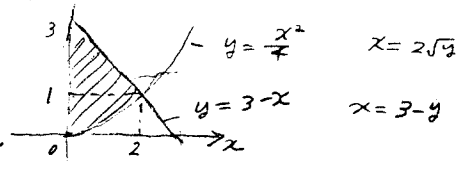
## §1. 重積分の計算

$$\begin{aligned}
 1.1 (1) \int_0^{\pi} \int_0^{\frac{\pi}{2}} x \sin(x+y) dx dy &= \int_0^{\pi} [-x \cos(x+y) + \sin(x+y)]_0^{\frac{\pi}{2}} dy \\
 &= \int_0^{\pi} \left(-\frac{\pi}{2} \cos\left(\frac{\pi}{2}+y\right) + \sin\left(\frac{\pi}{2}+y\right) - \sin y\right) dy \\
 &= \left[-\frac{\pi}{2} \sin\left(\frac{\pi}{2}+y\right) - \cos\left(\frac{\pi}{2}+y\right) + \cos y\right]_0^{\pi} \\
 &= \frac{\pi}{2} - 1 - (-\frac{\pi}{2} + 1) = \pi - 2
 \end{aligned}$$

$$\begin{aligned}
 (2) \int_0^1 dx \int_0^1 \sin^2 xy dy &= \int_0^1 dx \int_0^1 \frac{1 - \cos 2xy}{2} dy = \int_0^1 \left[\frac{y}{2} - \frac{\sin 2xy}{4x}\right]_0^1 dx \\
 &= \int_0^1 \left(\frac{1}{2}\right) dx = \frac{1}{2}
 \end{aligned}$$

$$(3) \int_0^2 dx \int_0^x xy^2 dy = \int_0^2 \left[\frac{x}{3} y^3\right]_0^x dx = \int_0^2 \frac{x^4}{3} dx = \frac{1}{3} \left[\frac{x^5}{5}\right]_0^2 = \frac{32}{15}$$

$$\begin{aligned}
 1.2 (1) \int_0^2 dx \int_{\frac{x^2}{4}}^{3-x} f(x,y) dy \\
 = \int_0^1 dy \int_0^{2\sqrt{y}} f(x,y) dx + \int_1^3 dy \int_0^{3-y} f(x,y) dx
 \end{aligned}$$



$$1.3 (1) \iint_D (x^2+y^2) dx dy \quad D: x^2+y^2 \leq 1, x+y \geq 1$$

$$= \int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} (x^2+y^2) dy$$

$$= \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_{1-x}^{\sqrt{1-x^2}} dx = \int_0^1 \left\{ x^2 \sqrt{1-x^2} + \frac{1}{3} (1-x^2)^{3/2} \sqrt{1-x^2} - \left[ x^2(1-x) + \frac{(1-x)^3}{3} \right] \right\} dx$$

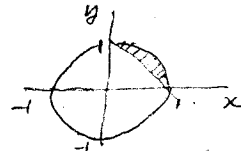
$$= \int_0^1 \left\{ \frac{1}{3} (1+2x^2) \sqrt{1-x^2} - \left( \frac{1}{3} - x + 2x^2 - \frac{4x^3}{3} \right) \right\} dx$$

$$= \frac{1}{3} \int_0^1 (1+2x^2) \sqrt{1-x^2} dx - \left[ \frac{1}{3} x - \frac{1}{2} x^2 + \frac{2}{3} x^3 - \frac{x^4}{3} \right]_0^1$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} (1+2\sin^2 t) \cos^3 t dt - \frac{1}{6}$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} (1 + \sin 2t - 2\sin^2 t) dt - \frac{1}{6}$$

$$= \frac{1}{3} \left( \frac{\pi}{2} + \frac{\pi}{4} - \frac{3}{2} \cdot \frac{\pi}{4} \right) - \frac{1}{6} = \frac{\pi}{8} - \frac{1}{6}$$


 $x = \sin t \quad t \in [0, \frac{\pi}{2}]$

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1.3 (2)  $\iint_D xy \, dx \, dy$

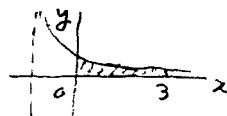
D:  $y \leq \sqrt{x}, y \geq 0, x \leq 1$



$$= \int_0^1 dx \int_0^{\sqrt{x}} xy \, dy = \int_0^1 \left[ \frac{x}{2} y^2 \right]_0^{\sqrt{x}} dx = \int_0^1 \frac{x^2}{2} dx = \left[ \frac{x^3}{6} \right]_0^1 = \frac{1}{6}$$

B1  $\iint_D (x+y^2) \, dx \, dy$

D:  $0 \leq x \leq 3, 0 \leq y \leq \frac{1}{x+1}$



$$\begin{aligned} &= \int_0^3 dx \int_0^{\frac{1}{x+1}} (x+y^2) dy = \int_0^3 \left[ xy + \frac{1}{3} y^3 \right]_0^{\frac{1}{x+1}} dx = \int_0^3 \left\{ \frac{x}{x+1} + \frac{1}{3} \left( \frac{1}{x+1} \right)^3 \right\} dx \\ &= \int_0^3 \left( 1 - \frac{1}{x+1} \right) dx + \frac{1}{3} \int_0^3 \frac{1}{(x+1)^3} dx = \left[ x - \log(x+1) - \frac{1}{2(x+1)^2} \right]_0^3 \\ &= 3 - \log 4 - \frac{1}{2 \cdot 4^2} + \frac{1}{2} = \frac{10}{2} - \log 4 \end{aligned}$$

(2)  $\iint_D (1+x+y) \, dx \, dy$

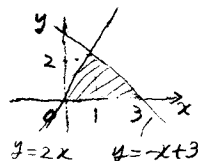
D:  $x \geq 0, y \geq 0, x+y \leq 1$



$$\begin{aligned} &= \int_0^1 dx \int_0^{1-x} (1+x+y) dy = \int_0^1 \left[ (1+x)y + \frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 \left\{ (1+x)(1-x) + \frac{(1-x)^2}{2} \right\} dx \\ &= \left[ x - \frac{x^3}{3} - \frac{(1-x)^3}{6} \right]_0^1 = 1 - \frac{1}{3} + \frac{1}{6} = \frac{5}{6} \end{aligned}$$

(5)  $\iint_D x\sqrt{y} \, dx \, dy$

D:  $y \leq 2x, y \leq -x+3, y \geq 0$

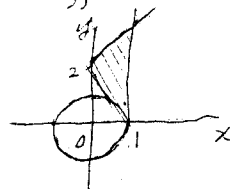


$$\begin{aligned} &= \int_0^2 dy \int_{\frac{y}{2}}^{3-y} x\sqrt{y} \, dx = \int_0^2 \left[ \frac{x^2}{2} \sqrt{y} \right]_{\frac{y}{2}}^{3-y} dy \\ &= \int_0^2 \left( \frac{(3-y)^2}{2} \sqrt{y} - \frac{y^2}{8} \sqrt{y} \right) dy = \int_0^{\sqrt{2}} \left( \frac{(3-x^2)^2}{2} \sqrt{x} - \frac{x^4}{8} \sqrt{x} \right) dx \quad \sqrt{y}=x, y=x^2 \\ &= \int_0^{\sqrt{2}} \left( \frac{9-x^2}{2} x^2 - \frac{1}{8} x^6 \right) dx = \int_0^{\sqrt{2}} \left( \frac{9}{2} x^2 - 6x^4 + x^6 - \frac{1}{8} x^6 \right) dx \\ &= \left[ \frac{3}{2} x^3 - \frac{6}{5} x^5 + \frac{7}{28} x^7 \right]_0^{\sqrt{2}} = 6\sqrt{2} - \frac{24}{5}\sqrt{2} + \frac{7 \cdot 8}{28}\sqrt{2} = \frac{72\sqrt{2}}{35} \end{aligned}$$

(6)  $\iint_D xy \, dx \, dy$

D:  $x^2 + y^2 \geq 1, 0 \leq x \leq 1$

$x - y + 2 \geq 0, y \geq 0$



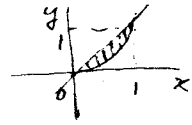
$$\begin{aligned} &= \int_0^1 dx \int_{\frac{x-2}{-1}}^{\sqrt{1-x^2}} xy \, dy \\ &= \int_0^1 x \left[ \frac{y^2}{2} \right]_{\frac{x-2}{-1}}^{\sqrt{1-x^2}} dx = \int_0^1 \left\{ \frac{x}{2} (1-x^2) - \frac{x}{2} (1-x)^2 \right\} dx \\ &= \int_0^1 \left( x^3 + 2x^2 + \frac{3}{2}x \right) dx = \left[ \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{3}{4}x^2 \right]_0^1 = \frac{5}{3} \end{aligned}$$

$$(7) \iint_D \frac{1}{\sqrt{1+x}} dx dy \quad D: 0 \leq x \leq 1, x^2 \leq y \leq x$$

$$= \int_0^1 dx \int_{x^2}^x \frac{1}{\sqrt{1+x}} dy = \int_0^1 \left( \frac{x}{\sqrt{1+x}} - \frac{x^2}{\sqrt{1+x}} \right) dx \quad \begin{array}{l} \sqrt{1+x} = t \\ x = t^2 - 1 \quad dx = 2t dt \end{array}$$

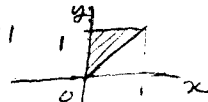
$$= \int_1^{\sqrt{2}} \left( \frac{t^2-1}{t} - \frac{(t^2-1)^2}{t} \right) 2t dt = 2 \int_1^{\sqrt{2}} (t^2-1-x^2+2x^2-1) dt$$

$$= 2 \left[ -\frac{t^5}{5} + t^3 - 2t \right]_1^{\sqrt{2}} = 2 \left( -\frac{2\sqrt{2}}{5} + 2\sqrt{2} - 2\sqrt{2} - \left( -\frac{1}{5} + 1 - 2 \right) \right) = \frac{12-8\sqrt{2}}{5}$$



$$(8) \iint_D x^2 y dx dy \quad D: x \geq 0, y \geq 0, 0 \leq x \leq y \leq 1$$

$$= \int_0^1 dy \int_0^y x^2 y dx = \int_0^1 \left[ \frac{x^3}{3} y^2 \right]_0^y dy = \int_0^1 \left( \frac{1}{3} y^5 - \frac{y^4}{3} \right) dy = \frac{1}{3} \left[ \frac{y^6}{6} - \frac{y^5}{5} \right]_0^1 = \frac{1}{15}$$



$$(9) \iint_D x dx dy \quad D: x^2 + y^2 \leq 4, x+y \leq 2$$

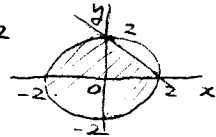
$$= \int_{-2}^0 x dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy + \int_0^2 x dx \int_{-\sqrt{4-x^2}}^{2-x} dy$$

$$= \int_{-2}^0 x \left[ y \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx + \int_0^2 x \left[ y \right]_{-\sqrt{4-x^2}}^{2-x} dx$$

$$= \int_{-2}^0 2x\sqrt{4-x^2} dx + \int_0^2 (x(2-x) + x\sqrt{1-x^2}) dx$$

$$= \left[ -\frac{2}{3} (4-x^2)^{\frac{3}{2}} \right]_{-2}^0 + \left[ x^2 - \frac{x^3}{3} \right]_0^2 - \left[ \frac{1}{3} (4-x^2)^{\frac{3}{2}} \right]_0^2$$

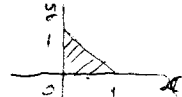
$$= -\frac{16}{3} + \frac{8}{3} + \frac{8}{3} = -\frac{8}{3}$$



$$(10) \iint_D (x^2 + y^2) dx dy \quad D: x \geq 0, y \geq 0, x+y \leq 1$$

$$= \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx = \int_0^1 \left( x^2(1-x) + \frac{(1-x)^3}{3} \right) dx$$

$$= \frac{1}{3} - \frac{1}{4} - \frac{1}{12} [(1-x)^4]_0^1 = \frac{1}{8}$$

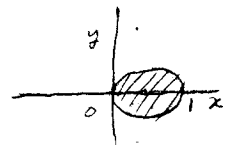


$$(11) \iint_D \sqrt{x} dx dy \quad D: x^2 + y^2 \leq x, \left(x - \frac{1}{2}\right)^2 + y^2 \leq \frac{1}{4}$$

$$= \int_0^1 \sqrt{x} dx \int_{-\sqrt{\frac{1}{4} - (x-\frac{1}{2})^2}}^{\sqrt{\frac{1}{4} - (x-\frac{1}{2})^2}} dy = \int_0^1 2\sqrt{x} \sqrt{x-x^2} dx = 2 \int_0^1 x \sqrt{1-x} dx$$

$$= 2 \int_1^0 (1-x^2) \sqrt{-x} dx = 4 \int_0^1 (x^2 - x^4) dx = 4 \left( \frac{1}{3} - \frac{1}{5} \right)$$

$$= \frac{8}{15}$$



$$\begin{array}{l} \sqrt{1-x} = t \\ 1-x^2 = x \end{array}$$

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$$1.4 (2) \iint_D (1-x^2-y^2) dx dy$$

$$= \int_0^{2\pi} d\theta \int_0^1 (1-r^2) r dr$$

$$= \int_0^{2\pi} \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$D: x^2+y^2 \leq 1 \quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$$



$$(3) \iint_D (x^2+y^2) dx dy \quad D: x^2+y^2 \leq a^2 \quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

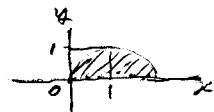
$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^a r^2 r dr = \int_0^{\frac{\pi}{2}} \frac{a^3}{4} d\theta = \frac{a^3 \pi}{8}$$

$$(4) \iint_D (1+x) dx dy \quad D: x^2+y^2 \leq 2x, y \geq 0$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} (1+r\cos\theta) r dr$$

$$= \int_0^{\frac{\pi}{2}} \left[ \frac{r^2}{2} + \frac{r^3}{3} \cos\theta \right]_0^{2\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} (2\cos^2\theta + \frac{8}{3}\cos^3\theta) d\theta$$

$$= 2 \frac{\pi}{4} + \frac{8}{3} \frac{3}{4} \frac{\pi}{2} = \pi \quad \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x dx = \frac{2n-1}{2n} \frac{2n-3}{2n-2} \dots \frac{1}{2} \frac{\pi}{2}$$



$$(5) \iint_D xy dx dy \quad D: x^2+y^2 \leq 1 \quad \begin{cases} x \geq 0 \\ y \geq 0 \end{cases} \quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^2 \sin\theta \cos\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin\theta \cos\theta d\theta = \frac{1}{8} [\sin^2\theta]_0^{\frac{\pi}{2}} = \frac{1}{8}$$

$$(6) \iint_D e^{-(x^2+y^2)} dx dy \quad D: x^2+y^2 \leq 1 \quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$$= \int_0^{2\pi} d\theta \int_0^1 e^{-r^2} r dr = \int_0^{2\pi} \left[ -\frac{1}{2} e^{-r^2} \right]_0^1 d\theta = \frac{1}{2} (1-e^{-1}) 2\pi = (1-e^{-1})\pi$$

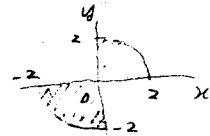
$$(7) \iint_D x dx dy \quad D: x^2+y^2 \leq 1 \quad x > 0 \quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^1 r^2 \cos\theta dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} \cos\theta d\theta = \frac{1}{3} [\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{3}$$

$$(8) \iint_D xy dx dy \quad D: x^2+y^2 \leq 4 \quad xy \geq 0 \quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$$= 2 \int_0^{\frac{\pi}{2}} d\theta \int_0^2 r^3 \sin\theta \cos\theta dr$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin\theta \cos\theta d\theta = 8 \left[ \frac{1}{2} \sin^2\theta \right]_0^{\frac{\pi}{2}} = 4$$



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$$(9) \iint_D y^2 dx dy \quad D: x^2 + y^2 \leq 1 \quad x = r \cos \theta, y = r \sin \theta, J = r$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3 \sin^2 \theta dr = \frac{1}{4} \int_0^{2\pi} \sin^2 \theta d\theta = \frac{1}{4} \cdot \pi = \frac{\pi}{4}$$

$$(10) \iint_D \left( \frac{x^3}{p} + \frac{y^2}{q} \right)^2 dx dy \quad D: x^2 + y^2 \leq a^2, p, q \text{ const}$$

$$x = r \cos \theta, y = r \sin \theta$$

$$= \int_0^{2\pi} d\theta \int_0^a \left( \frac{r^3 \cos^3 \theta}{p} + 2 \frac{r^4 \cos \theta \sin^2 \theta}{pq} + \frac{r^4 \sin^4 \theta}{q^2} \right) r dr$$

$$= \frac{a^6}{6} \int_0^{2\pi} \left( \frac{1}{p^2} \cos^6 \theta + \frac{2}{pq} (\cos^2 \theta - \cos^4 \theta) + \frac{1}{q^2} \sin^4 \theta \right) d\theta$$

$$= \frac{2}{3} a^6 \left( \frac{1}{p^2} \frac{3}{4} \frac{\pi}{4} + \frac{2}{pq} \frac{\pi}{4} - \frac{2}{pq} \frac{3}{4} \frac{\pi}{4} + \frac{1}{q^2} \frac{3}{4} \frac{\pi}{4} \right)$$

$$= \frac{2}{3} a^6 \left( \frac{1}{p^2} \cdot \frac{3}{16} + \frac{2}{16} \frac{1}{pq} + \frac{1}{8^2} \frac{3}{16} \right) \pi$$

$$= \left( \frac{1}{8p^2} + \frac{1}{12pq} + \frac{1}{8^2} \right) a^6 \pi$$

$$(11) \iint_D x^2 dx dy \quad D: x^2 + y^2 \leq 1 \quad x = r \cos \theta, y = r \sin \theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3 \cos^2 \theta dr = \frac{1}{4} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{1}{4} \cdot \pi = \frac{\pi}{4}$$

$$1.5 (1) \iint_D \sqrt{x^2 + y^2} dx dy \quad D: x^2 + y^2 \leq a^2, x^2 + y^2 - 2ax \geq 0$$

$$= 2 \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^a r^2 dr + 2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_{2a \cos \theta}^a r^2 dr \quad x = r \cos \theta, y = r \sin \theta$$

$$= \frac{2}{3} a^3 \frac{\pi}{2} + 2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3} (a^3 - 8a^3 \cos^3 \theta) d\theta \quad J = r$$

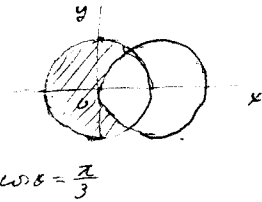
$$= \frac{4a^3 \pi}{9} - \frac{16}{3} a^3 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 - \cos^3 \theta) \cos \theta d\theta \quad r \leq a, r \geq 2a \cos \theta$$

$$= \frac{4a^3 \pi}{9} - \frac{16}{3} a^3 \left[ \sin \theta - \frac{1}{3} \sin^3 \theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{4a^3 \pi}{9} - \frac{16}{3} a^3 \left( \frac{2 - \sqrt{3}}{2} - \frac{1}{3} \frac{8 - 3\sqrt{3}}{8} \right) \quad a = 2a \cos \theta, \cos \theta = \frac{\pi}{3}$$

$$= \frac{4}{9} a^3 \pi - \frac{16}{3} a^3 \frac{(6 - 9\sqrt{3})}{24} = \frac{4}{9} a^3 \pi - \frac{32 - 18\sqrt{3}}{9} a^3$$

$$= \frac{2}{9} a^3 (2\pi - 16 + 9\sqrt{3})$$



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$$\begin{aligned}
 1.5(2) \quad & \iint_D \frac{\sqrt{1-x^2-y^2}}{1+x^2+y^2} dx dy \\
 &= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \frac{1-r}{1+r^2} r dr \\
 &= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 x \frac{(-2x)}{(1+x^2)^2} dx \\
 &= \frac{\pi}{2} \int_0^1 \frac{2x^2}{(1+x^2)^2} dx \\
 &= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \frac{2 \tan^2 \theta}{\sec^4 \theta} \sec^2 \theta d\theta \\
 &= \pi \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin^2 \theta d\theta \\
 &= \pi \int_0^{\frac{\pi}{4}} \left( \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta = \pi \left[ \frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{2} \left( \frac{\pi}{4} - \frac{1}{2} \right)
 \end{aligned}$$

$D: x^2 + y^2 \leq r^2, x \geq 0, y \geq 0$   
 $(r=1 \text{ の } \pi/4 \text{ 部分})$   
 $x = r \cos \theta, y = r \sin \theta, J = r$

$\sqrt{\frac{1-r}{1+r}} = x$  とおく  
 $1-r = (1+r^2)x^2 \quad r = \frac{1-x^2}{1+x^2}$   
 $2r dr = \frac{-2x(1+x^2) + 2x(1-x^2)}{(1+x^2)^2} dx$   
 $= \frac{-4x}{(1+x^2)^2} dx$   
 $x = \tan \theta \quad dx = \sec^2 \theta d\theta$   
 $\sin^2 \theta = \frac{1-\cos 2\theta}{2}$

(3)  $\iint_D \frac{1}{\sqrt{1-x^2-y^2}} dx dy$

$D: x^2 + y^2 \leq 1, x = r \cos \theta, y = r \sin \theta$

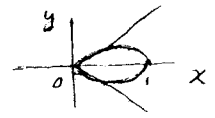
$= \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1-r^2}} dr = 2\pi \left[ -\sqrt{1-r^2} \right]_0^1 = 2\pi$

(4)  $\iint_D (1+x^2+y^2) dx dy$

$D: (x^2+y^2) \leq x^2+y^2, x \geq 0$

$x = r \cos \theta, y = r \sin \theta, J = r$

$r^4 \leq r^2 \cos^2 \theta$   
 $r^2 \leq \cos^2 \theta$



$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{\cos 2\theta}} (1+r^2) r dr$   
 $= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[ \frac{r^2}{2} + \frac{r^4}{4} \right]_0^{\sqrt{\cos 2\theta}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta \right) d\theta$   
 $= \left[ \frac{1}{4} \sin 2\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 \theta \cdot \frac{1}{2} d\theta$   
 $= \frac{\pi}{4} + \frac{1}{8} \cdot 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} + \frac{1}{8} \pi$

$2\theta = t$   
 $d\theta = \frac{1}{2} dt$

(5)  $\iint_D \frac{1}{\sqrt{1+x^2+y^2}} dx dy$

$D: x^2 + y^2 \leq 1, x = r \cos \theta, y = r \sin \theta$

$J = r$

$= \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1+r^2}} dr$   
 $= 2\pi \left[ \sqrt{1+r^2} \right]_0^1 = 2\pi(\sqrt{2} - 1)$

(6)  $\iint_D \sqrt{x^2+y^2} dx dy$

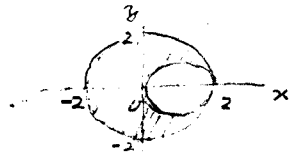
$D: (x-1)^2 + y^2 \leq 1, x^2 + y^2 \leq 2x$

$x = r \cos \theta, y = r \sin \theta, 0 \leq \theta < 2\pi$

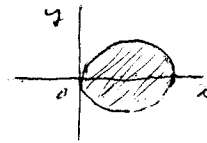
$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2 \cos \theta} r^2 dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3} \cos^3 \theta d\theta = \frac{8}{3} \cdot 2 \cdot \frac{2}{3} = \frac{32}{9}$

p. 62

$$\begin{aligned}
 (7) \quad & \iint_D \sqrt{x^2+y^2} \, dx \, dy \quad D: \quad 2x \leq x^2+y^2 \leq 4 \\
 & = 2 \left\{ \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^2 r^2 \, dr + \int_0^{\frac{\pi}{2}} d\theta \int_{2\cos\theta}^2 r^2 \, dr \right\} \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ 2\cos\theta &\leq r \leq 2 \end{aligned} \quad J=r \\
 & = 2 \left\{ \int_{\frac{\pi}{2}}^{\pi} \frac{r^3}{3} d\theta + \int_0^{\frac{\pi}{2}} \left( \frac{r^3}{3} - \frac{r^3}{3} \cos^3\theta \right) d\theta \right\} \\
 & = 2 \left\{ \frac{4}{3}\pi + \frac{4}{3}\pi - \frac{8}{3} \cdot \frac{\pi}{3} \right\} = \frac{16}{9} (3\pi - 2)
 \end{aligned}$$

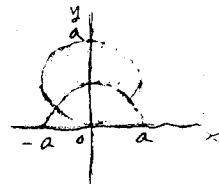


$$\begin{aligned}
 (8) \quad & \iint_D \sqrt{9-x^2-y^2} \, dx \, dy \quad D: \quad x^2+y^2 \leq 3x \\
 & = 2 \int_0^{\frac{\pi}{2}} d\theta \int_0^{3\cos\theta} \sqrt{9-r^2} \, r \, dr \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ 0 &\leq r \leq 3\cos\theta \\ -\frac{\pi}{2} &\leq \theta \leq \frac{\pi}{2} \end{aligned} \quad J=r \\
 & = 2 \int_0^{\frac{\pi}{2}} \left[ -\frac{1}{3} \sqrt{9-r^2} \right]_0^{3\cos\theta} d\theta \\
 & = \frac{2}{3} \int_0^{\frac{\pi}{2}} (27 - 27\sin^3\theta) d\theta = \frac{2}{3} (27 \cdot \frac{\pi}{2} - 27 \cdot \frac{2}{3}) = 9\pi - 12
 \end{aligned}$$



$$\begin{aligned}
 (9) \quad & \iint_D \frac{1}{\sqrt{1-x^2-y^2}} \, dx \, dy \quad D: \quad x^2+y^2 \leq 1 \\
 & = \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1-r^2}} \, dr \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \quad J=r \\
 & = \int_0^{2\pi} \left[ -\sqrt{1-r^2} \right]_0^1 d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad & \iint_D xy \, dx \, dy \quad D: \quad x^2+y^2 \leq a^2 \quad x \geq 0 \quad y \geq 0 \\
 & = \int_0^{\frac{\pi}{2}} d\theta \int_0^a r^3 \cos\theta \sin\theta \, dr \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ a &\leq r \leq a \end{aligned} \quad J=r \\
 & = \int_0^{\frac{\pi}{2}} \frac{a^4}{4} \sin\theta \cos\theta \, d\theta = \frac{a^4}{4} \left[ \frac{\sin^2\theta}{2} \right]_0^{\frac{\pi}{2}} = \frac{a^4}{8}
 \end{aligned}$$

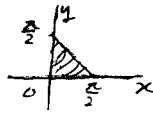


$$\begin{aligned}
 1.6 \quad & \iint_D |x| \, dx \, dy \quad D: \quad a \leq r \leq 2a \sin\theta \\
 & = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta \int_a^{2a\sin\theta} r^2 \cos\theta \, dr \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ a &= 2a\sin\theta \quad \theta = \frac{\pi}{2} \end{aligned} \\
 & = 2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta \int_a^{2a\sin\theta} r^2 \cos\theta \, dr = 2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{3} (8a^3 \sin^3\theta \cos\theta - a^3 \cos\theta) d\theta \\
 & = \frac{2}{3} \left[ 8a^3 \frac{1}{4} \sin^4\theta - a^3 \sin\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{2}{3} \left( 2a^3 - a^3 - \left( \frac{1}{8}a^3 - \frac{a^3}{2} \right) \right) \\
 & = \frac{11}{12} a^3
 \end{aligned}$$

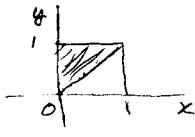
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1.6 (2)  $\iint_D |x y| dx dy$   $D: 2 \leq \sqrt{x^2 + y^2} \leq 3$   
 $x = r \cos \theta$   $J = r$   $2 \leq r \leq 3$   
 $y = r \sin \theta$   
 $= \int_0^{2\pi} d\theta \int_2^3 r^2 |\sin \theta \cos \theta| r dr$   
 $= 4 \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta \int_2^3 r^3 dr = 4 \left[ \frac{\sin^2 \theta}{2} \right]_0^{\frac{\pi}{2}} \left[ \frac{r^4}{4} \right]_2^3 = 2 \cdot \frac{81-16}{4} = \frac{65}{2}$

(3)  $\iint_D \sin(x+y) dx dy$   $D: x \geq 0, y \geq 0, x+y \leq \frac{\pi}{2}$   
 $= \int_0^{\frac{\pi}{2}} dx \int_0^{\frac{\pi}{2}-x} \sin(x+y) dy = \int_0^{\frac{\pi}{2}} [-\cos(x+y)]_0^{\frac{\pi}{2}-x} dx$   
 $= \int_0^{\frac{\pi}{2}} (\cos x - \cos \frac{\pi}{2}) dx = [\sin x]_0^{\frac{\pi}{2}} = 1$

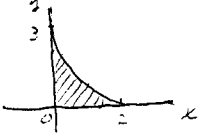


(4)  $\iint_D \cos x dx dy$   $D: 0 \leq x \leq y \leq 1$   
 $= \int_0^1 dx \int_x^1 \cos x dy = \int_0^1 [y \cos x]_x^1 dx = \int_0^1 (1-x) \cos x dx$   
 $= [\sin x - x \sin x - \cos x]_0^1 = 1 - \cos 1$

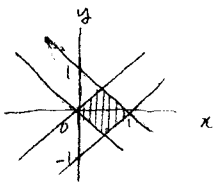


1.7  $\iint_D (x^2 + y^2) dx dy$   $D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$   
 $x = ar \cos \theta$   $J = ab r$   $0 \leq r \leq 1$   
 $y = br \sin \theta$   
 $= \int_0^{2\pi} d\theta \int_0^1 ab (a^2 \cos^2 \theta + b^2 \sin^2 \theta) r^3 dr$   
 $= \frac{ab}{4} \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \frac{ab}{4} (a^2 \pi + b^2 \pi) = \frac{ab\pi}{4} (a^2 + b^2)$

1.8 (1)  $\int_0^2 \int_0^{3+\frac{3}{2}x-3\sqrt{2x}} y dx dy$   $D: x \geq 0, y \geq 0, \sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}} \leq 1$   
 $\frac{y}{3} = (1 - \frac{\sqrt{x}}{\sqrt{2}})^2 = (1 - \sqrt{2x} + \frac{x}{2})$   
 $y = 3 + \frac{3}{2}x - 3\sqrt{2x}$   
 $= \int_0^2 \left[ \frac{1}{2} y^2 \right]_0^{3+\frac{3}{2}x-3\sqrt{2x}} dx = \frac{1}{2} \int_0^2 \left( \frac{9}{4} x^2 + 27x - 9\sqrt{2} x^{\frac{3}{2}} - 18\sqrt{2} x^{\frac{1}{2}} + 9 \right) dx$   
 $= \frac{1}{2} \left[ \frac{3}{4} x^3 + \frac{27}{2} x^2 - \frac{18\sqrt{2}}{5} x^{\frac{5}{2}} - 12\sqrt{2} x^{\frac{3}{2}} + 9x \right]_0^2$   
 $= \frac{1}{2} (6 + 54 - \frac{18 \times 8}{5} - 48 + 18) = \frac{3}{5}$



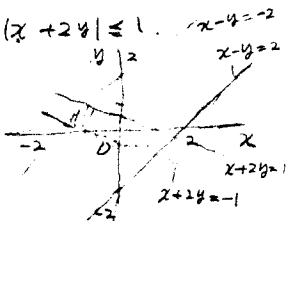
(2)  $\iint_D x dx dy$   $D: 0 \leq x+y \leq 1, 0 \leq x-y \leq 1$   
 $u = x+y$   $x = \frac{1}{2}(u+v)$   
 $v = x-y$   $y = \frac{1}{2}(u-v)$   
 $|J| = \frac{1}{2}$   
 $= \int_0^1 du \int_0^1 \frac{1}{2} (u+v) dv$   
 $= \frac{1}{4} \int_0^1 (u + \frac{1}{2}) du = \frac{1}{4}$





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$$\begin{aligned}
 (3) \quad & \iint_D \{(x-y)^2 + (x+2y)^2\} dx dy \\
 &= \int_{-2}^2 du \int_{-1}^1 (u^2 + v^2) \frac{1}{3} du dv \\
 &= \frac{1}{3} \int_{-2}^2 \left[ \frac{u^3}{3} + \frac{u^3}{3} \right]_{-1}^1 du = \frac{1}{3} \int_{-2}^2 (2u^2 + \frac{2}{3}) du \\
 &= \frac{1}{3} \left[ \frac{2}{3} u^3 + \frac{2}{3} u \right]_{-2}^2 \\
 &= \frac{1}{3} \left( \frac{8}{3} + \frac{4}{3} \right) = \frac{40}{9}
 \end{aligned}$$

$$\begin{aligned}
 D: \quad & |x-y| \leq 2 \quad |x+2y| \leq 1 \\
 & u = x-y \\
 & v = x+2y \\
 & x = \frac{2}{3}u + \frac{1}{3}v \\
 & y = -\frac{1}{3}u + \frac{1}{3}v \\
 & J = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}
 \end{aligned}$$


$$\begin{aligned}
 (4) \quad & \iint_D (x+y) e^{x-y} dx dy \\
 &= \int_0^1 du \int_0^1 u e^v \cdot \frac{1}{2} dv \\
 &= \frac{1}{2} \int_0^1 u (e-1) du = \frac{e-1}{4}
 \end{aligned}$$

$$\begin{aligned}
 D: \quad & 0 \leq x+y \leq 1 \\
 & 0 \leq x-y \leq 1 \\
 & u = x+y \quad x = \frac{1}{2}(u+v) \\
 & v = x-y \quad y = \frac{1}{2}(u-v) \\
 & |J| = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 1.9 \quad & \iiint_D (x+y+z) dx dy dz \quad D: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \\
 &= abc \iiint_D (au+bv+cw) du dv dw \\
 &= abc \iiint_{u^2+v^2 \leq 1} du dv \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} [(au+bv)w + \frac{c}{2}w^2] dw \\
 &= 2abc \iint_{u^2+v^2 \leq 1} (au+bv) \sqrt{1-u^2-v^2} du dv \\
 &= 2abc \int_0^{2\pi} (a \cos \theta + b \sin \theta) d\theta \int_0^1 \sqrt{1-r^2} r^2 dr \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 J &= abc \quad u^2+v^2+w^2 \leq 1 \\
 D': \quad & u^2+v^2+w^2 \leq 1 \\
 & \begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases} \quad J = r \\
 & 0 \leq r \leq 1
 \end{aligned}$$

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## § 2 重積分

$$2.1 \quad z = \log \frac{1}{r} = -\log r \quad r = \sqrt{x^2 + y^2}$$

$$= -\frac{1}{2} \log(x^2 + y^2)$$

$$(1) \quad z_x = \frac{-x}{x^2 + y^2} \quad z_{xx} = \frac{-(x^2 + y^2) + 2x^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$z_y = \frac{-y}{x^2 + y^2} \quad z_{yy} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\therefore z_{xx} + z_{yy} = 0$$

$$(2) \quad \int_A z \, dx \, dy \quad A = \{(x, y) \mid x^2 + y^2 \leq R^2, R > 0\} \quad (\leq R^2 \equiv x^2 + y^2)$$

$$= \iint_A -\frac{1}{2} \log(x^2 + y^2) \, dx \, dy \quad \begin{cases} x = r \cos \theta & r \geq 0 \\ y = r \sin \theta & 0 \leq \theta \leq 2\pi \end{cases} \quad J = r$$

$$= -\int_0^{2\pi} d\theta \int_0^R \log r \cdot r \, dr$$

$$= -\int_0^{2\pi} \left[ \frac{r^2}{2} \log r - \frac{r^2}{4} \right]_0^R d\theta = \int_0^{2\pi} \left( \frac{R^2}{4} - \frac{R^2}{2} \log R \right) d\theta = \frac{R^2 \pi}{2} - \pi R^2 \log R$$

2.2

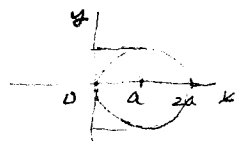
$$\iint_D |x y| \, dx \, dy \quad D: x^2 + y^2 \leq 2ax$$

$$= 2 \int_0^{\frac{\pi}{2}} d\theta \int_0^{2a \cos \theta} r^3 \sin \theta \cos \theta \, dr$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{16}{4} a^4 \cos^4 \theta \sin \theta \, d\theta$$

$$= 8a^4 \left[ -\frac{\cos^5 \theta}{5} \right]_0^{\frac{\pi}{2}} = \frac{8}{5} a^4$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ r \leq 2a \cos \theta \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{cases} \quad J = r$$

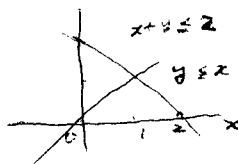


2.3

$$\iint_D z x y \, dx \, dy \quad D: \begin{cases} y \geq 0 \\ y \leq x \\ x + y \leq 2 \end{cases}$$

$$= \int_0^1 dy \int_y^{2-y} 2xy \, dx = \int_0^1 [x^2 y]_y^{2-y} dy$$

$$= \int_0^1 (\omega - y)^2 y - y^3 dy = \left[ \frac{y^4}{4} - \frac{4y^3}{3} + \frac{4}{2} y^2 - \frac{y^4}{4} \right]_0^1 = \frac{2}{3}$$



2.4

$$(1) \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{matrix} y \\ r \\ \theta \\ x \end{matrix} \quad |J| = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \left| \begin{matrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{matrix} \right| = r$$

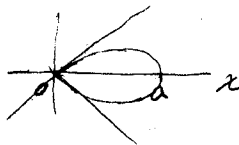
$$(2) \quad \iint_D \sqrt{a^2 - x^2 - y^2} \, dx \, dy \quad D: x^2 + y^2 \leq a^2$$

$$= \int_0^{2\pi} d\theta \int_0^a \sqrt{a^2 - r^2} \cdot r \, dr = \int_0^{2\pi} \left[ -\frac{1}{3} (a^2 - r^2)^{\frac{3}{2}} \right]_0^a d\theta = \frac{a^3}{3} \int_0^{2\pi} d\theta = \frac{2a^3 \pi}{3}$$

P.62, P.63

2.5, 2.31 = 171'

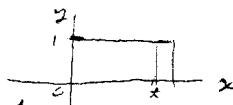
2.6 (1)  $r = a \cos 2\theta \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$



$$\begin{aligned}
 (2) \quad & \iint_D x \, dx \, dy && x = r \cos \theta \quad y = r \sin \theta \\
 & = \int_{-\pi/4}^{\pi/4} d\theta \int_0^{a \cos 2\theta} r^2 \cos \theta \, dr && J = r \\
 & = \int_{-\pi/4}^{\pi/4} \frac{1}{3} a^3 \cos^3 2\theta \cos \theta \, d\theta = \frac{a^3}{3} \int_{-\pi/4}^{\pi/4} (1 - 2 \sin^2 \theta)^2 \cos \theta \, d\theta \\
 & = \frac{a^3}{3} \int_{-\pi/4}^{\pi/4} (8 \sin^6 \theta - 12 \sin^4 \theta + 6 \sin^2 \theta - 1) \cos \theta \, d\theta \\
 & = -\frac{a^3}{3} \left[ \frac{8}{7} \sin^7 \theta - \frac{12}{5} \sin^5 \theta + 2 \sin^3 \theta - \sin \theta \right]_{-\pi/4}^{\pi/4} \\
 & = -\frac{2a^3}{3} \left( \frac{1}{\sqrt{2}} - \frac{3}{5\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) = \frac{-2a^3}{3\sqrt{2}} \left( \frac{1}{5} - \frac{3}{5} \right) \\
 & = \frac{2a^3}{3\sqrt{2}} \frac{16}{35} = \frac{16\sqrt{2}}{105} a^3
 \end{aligned}$$

2.7  $F(x) = \iint_{I_x} f(x, y) \, dx \, dy \quad I_x = \{ (x, y) \mid 0 \leq x \leq t, 0 \leq y \leq 1 \}$

$$= \int_0^1 dy \int_0^x f(x, y) \, dx$$



$$\begin{aligned}
 F(x+t) - F(x) &= \int_0^1 \left\{ \int_0^{x+t} f(x, y) \, dx - \int_0^x f(x, y) \, dx \right\} dy \\
 &= \int_0^1 \left\{ \int_x^{x+t} f(x, y) \, dx \right\} dy = \int_x^{x+t} \left\{ \int_0^1 f(x, y) \, dy \right\} dx \\
 \therefore F(x) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \{ F(x+\Delta x) - F(x) \} = \int_0^1 f(x, y) \, dy
 \end{aligned}$$

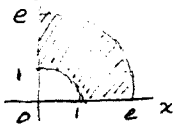
2.8 (1)  $D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \quad u = \frac{x}{a} \quad x = au \quad J = ab$

$$\begin{aligned}
 & \iint_D f(x, y) \, dx \, dy && v = \frac{y}{b} \quad y = bv \\
 & = \iint_{D'} ab f(au, bv) \, du \, dv && D': u^2 + v^2 \leq 1
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \iint_D (x^2 + y^2) \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} \, dx \, dy = \iint_{D'} ab(a^2 u^2 + b^2 v^2) \sqrt{u^2 + v^2} \, du \, dv \\
 & = \int_0^{2\pi} d\theta \int_0^1 ab(b^2 \cos^2 \theta + a^2 \sin^2 \theta) r^2 \, dr && u = r \cos \theta \quad J = r \\
 & = \frac{ab}{5} \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \, d\theta = \frac{2ab\pi}{5} (a^2 + b^2)
 \end{aligned}$$

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2.9 (1)  $D: x = e^u \cos v, y = e^u \sin v, 0 \leq u \leq 1, 0 \leq v \leq \frac{\pi}{2}$



$e^{2u} = x^2 + y^2$   
 $\tan v = \frac{y}{x}$   
 $u = \frac{1}{2} \log(x^2 + y^2)$   
 $v = \tan^{-1} \frac{y}{x}$

(2)  $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}$

(3)  $\iint_D xy \, dx \, dy = \int_0^{\frac{\pi}{2}} \int_0^1 e^{2u} \cos v \sin v e^{2u} \, du \, dv$   
 $= \int_0^{\frac{\pi}{2}} \cos v \sin v \, dv \int_0^1 e^{4u} \, du = \frac{1}{2} [\sin^2 v]_0^{\frac{\pi}{2}} \cdot \left[ \frac{1}{4} e^{4u} \right]_0^1$   
 $= \frac{1}{8} (e^4 - 1)$

2.10  $I = \iint_D \frac{1-x^2-y^2}{1+x^2+y^2} \, dx \, dy, D: x^2 + y^2 \leq 1, x \geq 0, y \geq 0$

(1)  $x = r \cos \theta, y = r \sin \theta, J = r, D': 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}$

$I = \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1-r^2}{1+r^2} r \, dr \, d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \left( -1 + \frac{2}{1+r^2} \right) r \, dr$

(2)  $I = \int_0^{\frac{\pi}{2}} \left[ -\frac{r^2}{2} + \log(1+r^2) \right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \left( -\frac{1}{2} + \log 2 \right) d\theta$   
 $= \frac{\pi}{2} \left( \log 2 - \frac{1}{2} \right)$

2.11  $I = \iint_D \max(x^2, y) \, dx \, dy, D: 0 \leq x, 0 \leq y, x+y \leq 1$

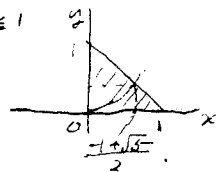
$= \int_0^{\frac{1+\sqrt{5}}{2}} dx \int_0^{1-x} y \, dy + \int_0^{\frac{1+\sqrt{5}}{2}} dx \int_0^{x^2} x^2 \, dy$

$+ \int_{\frac{1+\sqrt{5}}{2}}^1 dx \int_0^{1-x} x^2 \, dy$

$= \int_0^{\frac{1+\sqrt{5}}{2}} \frac{1}{2} (1-x)^2 x^2 \, dx + \int_0^{\frac{1+\sqrt{5}}{2}} x^4 \, dx + \int_{\frac{1+\sqrt{5}}{2}}^1 x^2 (1-x) \, dx$

$= \frac{1}{2} \left[ -\frac{(1-x)^3}{3} - \frac{x^5}{5} \right]_0^{\frac{1+\sqrt{5}}{2}} + \left[ \frac{x^5}{5} \right]_0^{\frac{1+\sqrt{5}}{2}} + \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_{\frac{1+\sqrt{5}}{2}}^1$

$= (25\sqrt{5} - 31) / 120$

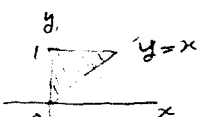


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$$12, \iint_D (x^2 - y) dx dy \quad 0 \leq x \leq y \leq 1$$

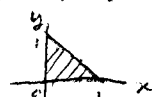
$$= \int_0^1 dx \int_x^1 (x^2 - y) dy = \int_0^1 \left[ x^2 y - \frac{y^2}{2} \right]_x^1 dx = \int_0^1 \left( x^2 - \frac{1}{2} - x^2 + \frac{x^2}{2} \right) dx$$

$$= \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} = -\frac{1}{4}$$

$$2.13 \iint_D \frac{x}{\sqrt{x^2 + y^2}} dx dy \quad D: \text{triangle with vertices } (0,0), (1,0), (0,1)$$


$$= \int_0^1 dy \int_0^y \frac{x}{\sqrt{x^2 + y^2}} dx$$

$$= \int_0^1 \left[ \sqrt{x^2 + y^2} \right]_0^y dy = \int_0^1 (\sqrt{2}y - y) dy = \frac{\sqrt{2}-1}{2} [y^2]_0^1 = \frac{\sqrt{2}-1}{2}$$

$$2.14 \iint_D (x^2 + y^2) dx dy \quad D: x \geq 0, y \geq 0, x+y \leq 1$$


$$= \int_0^1 dx \int_0^{1-x} (x^2 + y^2) dy = \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left( x^2(1-x) + \frac{1}{3}(1-x)^3 \right) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} - \frac{1}{12}(1-x)^4 \right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}$$

$$2.15 \iint_D (x+y) e^{x-y} dx dy \quad D: 0 \leq x+y \leq 1, 0 \leq x-y \leq 1$$

$$u = x+y, \quad x = \frac{1}{2}(u+v)$$

$$v = x-y, \quad y = \frac{1}{2}(u-v)$$

$$J = \frac{1}{2}$$

$$= \frac{1}{2} \int_0^1 u du \int_0^u e^v dv$$

$$= \frac{1}{4}(e-1)$$

$$2.16 \iint_D \frac{\log(x^2 + y^2)}{x^2 + y^2} dx dy \quad D: x \geq 0, y \geq 0, 1 \leq x^2 + y^2 \leq 4$$

$$x = r \cos \theta, \quad r = r$$

$$y = r \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$1 \leq r \leq 2$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_1^2 \frac{2 \log r}{r^2} r dr$$

$$= \frac{\pi}{2} \left[ (\log r)^2 \right]_1^2 = \frac{\pi}{2} (\log 2)^2$$

$$2.17 \iint_D x^2 \exp(x^2 + y^2) dx dy \quad D: x^2 + y^2 \leq 4, x \geq 0$$

$$x = r \cos \theta, \quad r = r$$

$$y = r \sin \theta, \quad 0 \leq \theta \leq \pi$$

$$0 \leq r \leq 2$$

$$= \int_0^{\pi} \cos^2 \theta d\theta \int_0^2 r^3 e^{r^2} dr$$

$$= \frac{\pi}{2} \left\{ \left[ \frac{r^2 e^{r^2}}{2} \right]_0^2 - \int_0^2 r e^{r^2} dr \right\}$$

$$= \frac{\pi}{2} \left\{ 2e^4 - \frac{1}{2} [e^{r^2}]_0^2 \right\} = \frac{3}{4} (e^4 - 1)$$

### § 3 広義積分

3.1 (1)  $\iint_D \frac{x+y}{x^2+y^2} dx dy$      $D: 0 \leq x \leq 1, 0 \leq y \leq x$

$$= \int_0^1 dx \int_0^x \left( \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} \right) dy = \int_0^1 \left[ \tan^{-1} \frac{y}{x} + \frac{1}{2} \log(x^2+y^2) \right]_0^x dx$$

$$= \int_0^1 \left[ \frac{\pi}{4} + \frac{1}{2} (\log 2x^2 - \log x^2) \right] dx = \frac{\pi}{4} + \frac{1}{2} \int_0^1 (\log 2) dx = \frac{\pi}{4} + \frac{1}{2} \log 2$$

(2)  $\iint \frac{1}{(x+y)^2} dx dy$      $1/(x+y)^2 \rightarrow (x+y)^{-2}$      $\int u^{-2} du = -1/u$

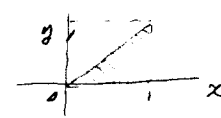
$$= \int_0^1 dx \int_0^1 (x+y)^{-2} dy$$

$D: 0 \leq x \leq 1, 0 \leq y \leq 1$

$$= \int_0^1 \left[ -2(x+y)^{-1} \right]_0^1 dx = \int_0^1 \left\{ -2(x+1)^{-1} - (-2x^{-1}) \right\} dx = \left[ -2 \log(x+1) + 2 \log x \right]_0^1$$

$$= 2 - 4 \log 2 = 2 - 4 \log 2$$

(3)  $\iint_D \frac{x}{\sqrt{x^2+y^2}} dx dy$      $D: 0 \leq x \leq 1, 0 \leq y \leq x$

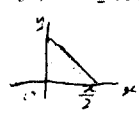
$$= \int_0^1 dy \int_y^1 \frac{x}{\sqrt{x^2+y^2}} dx = \int_0^1 \left[ \sqrt{x^2+y^2} \right]_y^1 dy$$


$$= \int_0^1 (\sqrt{1+y^2} - \sqrt{2y^2}) dy$$

$$= -\sqrt{2} \left[ \frac{1}{2} y^2 \right]_0^1 + \frac{1}{2} \left[ y\sqrt{y^2+1} + \log(y+\sqrt{y^2+1}) \right]_0^1$$

$$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} \log(1+\sqrt{2}) = \frac{1}{2} \log(1+\sqrt{2})$$

3.2 (1)  $\iint_D \sin^{-1} \frac{2xy}{x^2+y^2} dx dy$      $D: 0 \leq x, 0 \leq y, 0 \leq x+y \leq \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi/2\theta} 2\theta r dr$$


$$= \int_0^{\frac{\pi}{2}} \left[ \theta r^2 \right]_0^{\pi/2\theta} d\theta = \int_0^{\frac{\pi}{2}} \theta \frac{\pi^2}{4} d\theta = \frac{\pi^2}{8} \int_0^{\frac{\pi}{2}} \theta d\theta = \frac{\pi^2}{8} \left[ \frac{\theta^2}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8} \cdot \frac{\pi^2}{8} = \frac{\pi^4}{64}$$

$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1} \frac{y}{x} \end{cases} \quad 0 \leq \theta \leq \frac{\pi}{2}$   
 $\frac{2xy}{x^2+y^2} = 2 \cos \theta \sin \theta = \sin 2\theta$   
 $\sin^{-1} \frac{2xy}{x^2+y^2} = 2\theta$   
 $\sqrt{2} r \sin(\theta + \frac{\pi}{2}) \leq \frac{\pi}{2}$   
 $r \leq \frac{\pi}{2\sqrt{2} \sin(\theta + \frac{\pi}{2})}$

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$$\begin{aligned}
 (2) \quad & \iint_D \cos \frac{x-y}{x+y} dx dy \\
 &= \int_0^1 du \int_{-u}^u \cos \frac{v}{u} dv \\
 &= \int_0^1 [x \sin \frac{v}{u}]_{-u}^u du \\
 &= 2 \int_0^1 u \sin 1 du = 2 \left[ \frac{u^2}{2} \right]_0^1 \sin 1 \\
 &= \sin 1
 \end{aligned}$$

D:  $x \geq 0, y \geq 0, 0 \leq x+y \leq 1$

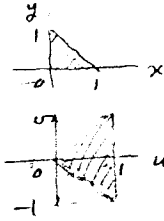
$u = x+y$

$v = x-y$

$x = \frac{1}{2}(u+v)$

$y = \frac{1}{2}(u-v)$

$|J| = \frac{1}{2}$



$$\begin{aligned}
 3.3 \quad (1) \quad & \iint_D e^{-(x^2+y^2)} dx dy \\
 &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} r e^{-r^2} dr \\
 &= \int_0^{\frac{\pi}{2}} \left[ -\frac{1}{2} e^{-r^2} \right]_0^{\infty} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}
 \end{aligned}$$

D:  $x \geq 0, y \geq 0$

$x = r \cos \theta, y = r \sin \theta$

$0 \leq r < \infty, 0 \leq \theta \leq \frac{\pi}{2}$

$J = r$

$$\begin{aligned}
 (2) \quad & \iint_0^{2\pi} \int_1^{\infty} \frac{1}{(x^2+y^2)^2} dx dy \\
 &= \int_0^{2\pi} d\theta \int_1^{\infty} \frac{1}{r^2} r dr \\
 &= \int_0^{2\pi} \left[ -\frac{1}{2r} \right]_1^{\infty} d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi
 \end{aligned}$$

D:  $x^2+y^2 \geq 1$

$x = r \cos \theta, y = r \sin \theta$

$J = r$

$r \geq 1, 0 \leq \theta \leq 2\pi$

$$\begin{aligned}
 3.4 \quad & \iint_D \frac{1}{(x^2+y^2)^p} dx dy \\
 &= \int_0^{2\pi} d\theta \int_0^a \frac{1}{r^{2p}} r dr \\
 &= \int_0^{2\pi} \left[ \frac{1}{2-2p} r^{2-2p} \right]_0^a d\theta \quad \because 2-2p > 0 \text{ のとき存在する} \\
 &= \frac{2\pi}{2(1-p)} a^{2(1-p)} = \frac{\pi}{1-p} a^{2(1-p)} \quad 0 < p < 1
 \end{aligned}$$

D:  $x^2+y^2 \leq a^2$

$x = r \cos \theta, y = r \sin \theta$

$J = r$

$0 \leq r \leq a, 0 \leq \theta \leq 2\pi$

$0 < p < 1$

$p > 1$  のときは存在しない

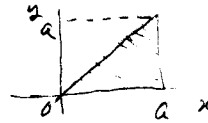
$$3.5 \quad (1) \quad \int \sin^{-1} \frac{x}{a} dx = x \sin^{-1} \frac{x}{a} - \int x \frac{1}{\sqrt{a^2-x^2}} dx = x \sin^{-1} \frac{x}{a} + \sqrt{a^2-x^2}$$

$$\begin{aligned}
 (2) \quad & \int_0^a dx \int_0^x \frac{f(y)}{\sqrt{(a-x)(x-y)}} dy \\
 &= \int_0^a dy \int_y^a \frac{f(y)}{a-x} \frac{1}{\sqrt{\frac{x-y}{a-x}}} dx \\
 &= \int_0^a dy \int_0^{\frac{a-y}{1+x}} \frac{f(y)}{1+x} \frac{1}{\sqrt{\frac{(a-y)x}{1+x^2}}} dx = \int_0^a dy \int_0^{\frac{\pi}{2}} \frac{f(y)}{1+x^2} dx = \int_0^a [\tan^{-1} x]_0^{\frac{\pi}{2}} f(y) dy \\
 &= \frac{\pi}{2} [f(y)]_0^a = \frac{\pi}{2} (f(a) - f(0))
 \end{aligned}$$

$\sqrt{\frac{x-y}{a-x}} = t$

$x = \frac{a(x-y)}{1+x^2}$

$\frac{dx}{dt} = \frac{(a-y)t}{(1+x^2)^2}$



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$$\S. 4. \int_0^{\infty} \exp(-x^2) dx$$

$$4.1 \quad (1) \int_0^{\infty} x^2 e^{-x^2} dx = \left[ x \left( -\frac{1}{2} e^{-x^2} \right) \right]_0^{\infty} + \int_0^{\infty} \frac{1}{2} e^{-x^2} dx = \frac{1}{2} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

$$(2) \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} r e^{-r^2} dr \quad \begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array} \quad J=r$$

$$= \frac{\pi}{2} \left[ -\frac{1}{2} e^{-r^2} \right]_0^{\infty} = \frac{\pi}{4}$$

$$(3) I = \int_0^{\infty} e^{-x^2} dx \quad k < k < k$$

$$I^2 = \int_0^{\infty} \int_0^{\infty} e^{-x^2} e^{-y^2} dx dy = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy = \frac{\pi}{4}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}$$

$$(4) \int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$(5) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2+y^2) e^{-(x^2+y^2)} dx dy \quad \begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array} \quad J=r$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r^3 e^{-r^2} dr = 2\pi \left\{ \left[ -\frac{r^2}{2} e^{-r^2} \right]_0^{\infty} + \int_0^{\infty} r e^{-r^2} dr \right\}$$

$$= 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_0^{\infty} = \pi$$

$$4.2 \quad (1) \int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \quad 4.1 \quad (1) = \frac{\sqrt{\pi}}{4}$$

$$(2) I = \int_0^{\infty} e^{-x^2} dx \quad k < k < k \quad I^2 = \int_0^{\infty} \int_0^{\infty} e^{-x^2} e^{-y^2} dx dy = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} r e^{-r^2} dr = \frac{\pi}{2} \left[ -\frac{1}{2} e^{-r^2} \right]_0^{\infty} = \frac{\pi}{4} \quad \begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array} \quad \begin{array}{l} k < k < k \\ J=r \end{array}$$

$$I = \frac{\sqrt{\pi}}{2}$$

$$4.3 \quad (1) \operatorname{Erf} x = \int_0^x e^{-t^2} dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1} \quad \text{收敛半径 } \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{(2n+2)(n+1)!}{(2n+1)n!} = \infty$$

\(\therefore\) 收敛半径 \(\infty\)

$$(2) \operatorname{Erf} (1) = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!}$$

$$= 1 - 0.3333 + 0.1 - 0.023 = 0.7$$

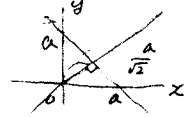


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$$4.4 \quad z = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \quad x+y \leq a$$

$$(1) \quad V(a) = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy - \int_D e^{-\frac{x^2+y^2}{2}} dx dy \right\} \quad D: x+y \geq a$$

$$(2) \quad V(a) = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy - \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \int_{\frac{a-x}{2}}^{\infty} e^{-\frac{y^2}{2}} dy \right\}$$



$$= \frac{1}{2\pi} \left\{ 2\sqrt{2}\sqrt{2} - \sqrt{2}\pi \int_{\frac{a}{2}}^{\infty} e^{-\frac{y^2}{2}} dy \right\}$$

$$\frac{x}{\sqrt{2}} = t, \quad dx = \sqrt{2} dt$$

$$= 1 + \frac{1}{\sqrt{2\pi}} \int_0^{\frac{a}{\sqrt{2}}} e^{-\frac{y^2}{2}} dy = 1 + \frac{1}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2}}} e^{-t^2} dt$$

$$\frac{y}{\sqrt{2}} = t$$

$$\therefore \frac{dV(a)}{da} = \frac{1}{2\sqrt{\pi}} e^{-\frac{a^2}{2}}$$

$$4.5 \quad \int \int_D e^{-(x^2+y^2)} dx dy$$

$$D: x \geq 0, y \geq 0$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} r e^{-r^2} dr$$

$$x = r \cos \theta, \quad y = r \sin \theta \quad J = r$$

$$= \frac{\pi}{2} \left[ -\frac{1}{2} e^{-r^2} \right]_0^{\infty} = \frac{\pi}{4}$$

$$\therefore (1) \quad I(R) = \int \int_D e^{-(x^2+y^2)} dS$$

$$D: x \geq 0, y \geq 0, x^2+y^2 \leq R$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^R r e^{-r^2} dr$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$J = r$$

$$= \frac{\pi}{2} \left[ -\frac{1}{2} e^{-r^2} \right]_0^R$$

$$0 \leq r \leq R, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

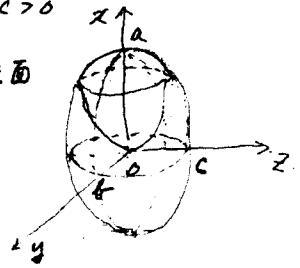
$$= \frac{\pi}{4} (1 - e^{-R^2})$$

$$(2) \quad I(R) < \int \int_D e^{-x^2-y^2} dx dy < I(\sqrt{2}R)$$

§ 5. 重積分の応用 (体積)

5.1 (1).  $\iint_D \tan^{-1} \frac{y}{x} dx dy$       $x=0, y=0, x^2+y^2=1, \theta=1$   
 $= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r dr = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \frac{\pi^2}{16}$       $x=r\cos\theta, y=r\sin\theta, J=r, \frac{y}{x} = \tan\theta$

(2)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}, a>0, b>0, c>0$   
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$       $\therefore$  "A" 体,  $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}$ . 平面  
 $\frac{x^2}{a^2} + \frac{x}{a} = 1 \quad x^2 + ax - a^2 = 0 \quad x = \frac{-1 \pm \sqrt{5}}{2} a$   
 $x > 0 \leq 1 \quad x = \frac{\sqrt{5}-1}{2} a$



$\therefore \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{\sqrt{5}-1}{2} \quad x = a\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}$   
 $V = \iint_D \{a\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}} - a(\frac{y^2}{b^2} + \frac{z^2}{c^2})\} dy dz$       $D: \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq \frac{\sqrt{5}-1}{2}$   
 $= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} A(\sqrt{1-r^2} - r^2) b c r dr$       $\begin{cases} y = br\cos\theta & b \leq c \\ z = cr\sin\theta & c \leq b \end{cases}$   
 $= abc \cdot [0]_0^{2\pi} \left[ -\frac{1}{3}(1-r^2)^{\frac{3}{2}} - \frac{r^4}{4} \right]_0^{\sqrt{a}}$       $J = bcr \quad 0 \leq r \leq 1$   
 $= 2abc\pi \frac{15-5\sqrt{5}}{24} = \frac{5}{12}(3-\sqrt{5})abc\pi$       $\Delta = \sqrt{\frac{\sqrt{5}-1}{2}} \quad c \leq b \leq c$

(3)  $x^2+z^2=a^2, y^2-z^2=b^2, x \geq 0, y \geq 0, z \geq 0$

$V = \iint_D \sqrt{a^2-z^2} dx dz$       $D: x^2+z^2 \leq a^2, x \geq 0, z \geq 0$   
 $= \int_0^a dz \int_0^{\sqrt{a^2-z^2}} \sqrt{a^2-z^2} dx = \int_0^a \frac{1}{2} [z\sqrt{a^2-z^2} + a^2 \sin^{-1} \frac{z}{a}]_0^{\sqrt{a^2-z^2}} dz$   
 $= \frac{1}{2} \int_0^a [z\sqrt{a^2-z^2} + a^2 \sin^{-1} \frac{\sqrt{a^2-z^2}}{a}] dz$   
 $= \frac{1}{2} [-\frac{1}{3}\sqrt{a^2-z^2}^3 + a^2 x \sin^{-1} \frac{\sqrt{a^2-z^2}}{a}]_0^a - \frac{a^2}{2} \int_0^a \frac{1}{x} \frac{-x}{\sqrt{a^2-x^2}} dx$   
 $= \frac{1}{6} a^3 + \frac{a^2}{2} \int_0^a \frac{x}{\sqrt{a^2-x^2}} dx = \frac{1}{6} a^3 + \frac{a^2}{2} [-\sqrt{a^2-x^2}]_0^a = \frac{1}{6} a^3 + \frac{1}{2} a^2 - \frac{2}{3} a^3$

(4)  $z = x^2 + y^2, z = y$

$D: x^2 + y^2 - y \leq 0$

$V = \iint_D (y - x^2 + y^2) dx dy$

$x^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}$

$= \int_0^{\pi} d\theta \int_0^{\sin\theta} (r\cos\theta - r^2) r dr$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad J = r \quad r \leq \sin\theta$

$= \int_0^{\pi} [\frac{r^3}{3} \cos\theta - \frac{r^4}{4}]_0^{\sin\theta} d\theta = \frac{1}{12} \int_0^{\pi} \sin^4\theta d\theta = \frac{1}{12} \cdot \frac{3}{4} \cdot \frac{\pi}{4} = \frac{\pi}{32}$

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$$(5) \quad z = x^2 + y^2, \quad z = 2x + 2y + 1 \quad x^2 + y^2 - 2x - 2y - 1 \leq 0$$

$$V = \iint_D (2x + 2y + 1 - x^2 - y^2) dx dy$$

$$D: (x-1)^2 + (y-1)^2 \leq 3$$

$$= \iint_D \{3 - (x-1)^2 - (y-1)^2\} dx dy$$

$$\begin{cases} x-1 = r \cos \theta \\ y-1 = r \sin \theta \end{cases} \quad J = r$$

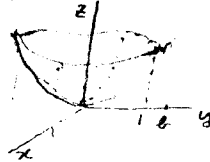
$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \{3 - r^2\} r dr = 2\pi \left[ \frac{3}{2} r^2 - \frac{1}{4} r^4 \right]_0^{\sqrt{3}} = \frac{9}{2} \pi$$

$$(6) \quad z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \quad x^2 + y^2 = 1, \quad z = 0$$

$$V = \iint_D \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) dx dy \quad D: x^2 + y^2 \leq 1$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left( \frac{r^2 \cos^2 \theta}{a^2} + \frac{r^2 \sin^2 \theta}{b^2} \right) r^2 dr \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad J = r$$

$$= \frac{1}{4} \int_0^{2\pi} \left( \frac{1}{a^2} \cos^2 \theta + \frac{1}{b^2} \sin^2 \theta \right) d\theta = \frac{1}{4} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \pi$$



$$(7) \quad x^2 + y^2 = a^2, \quad y^2 + z^2 = a^2, \quad z^2 + x^2 = a^2$$

$$V = 16 \iint_D \sqrt{a^2 - x^2} dx dy \quad D: \begin{cases} x^2 + y^2 \leq a^2 \\ 0 \leq y \leq x \end{cases}$$

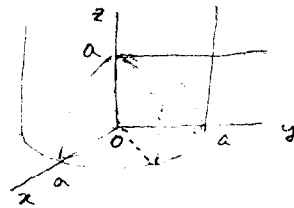
$$= 16 \int_0^{\frac{\pi}{4}} d\theta \int_0^a \sqrt{a^2 - r^2 \cos^2 \theta} r dr \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad J = r$$

$$= 16 \int_0^{\frac{\pi}{4}} \left[ -\frac{1}{3 \cos^3 \theta} (a^2 - r^2 \cos^2 \theta)^{\frac{3}{2}} \right]_0^a d\theta \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq \frac{\pi}{4}$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{4}} \left( \frac{a^3}{\cos^3 \theta} - \frac{a^3 \cos^3 \theta}{\cos^3 \theta} \right) d\theta = \frac{16}{3} a^3 \int_0^{\frac{\pi}{4}} \left( \sec^3 \theta + \cos \theta - \frac{\sin \theta}{\cos^2 \theta} \right) d\theta$$

$$= \frac{16}{3} a^3 \left[ \tan \theta - \cos \theta - \frac{1}{\cos \theta} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{16}{3} a^3 \left( 1 - \frac{1}{\sqrt{2}} + 1 - \sqrt{2} + 1 \right) = \frac{16}{3} a^3 \left( 3 - \frac{3\sqrt{2}}{2} \right) = 8a^3 (2 - \sqrt{2})$$



$$(8) \quad x = y^2, \quad z = a^2 - x^2, \quad x^2 + y^2 = \frac{a^2}{4}$$

$$-V = \iint_D \sqrt{2(a^2 - x^2 - y^2)} dx dy \quad D: x^2 + y^2 \leq \frac{a^2}{4}$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{a}{2}} \sqrt{2(a^2 - r^2)} r dr$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad J = r$$

$$0 \leq r \leq \frac{a}{2}$$

$$0 \leq \theta \leq 2\pi$$

$$= 2\pi \left[ 2 \left( \frac{1}{3} (a^2 - r^2)^{\frac{3}{2}} \right) \right]_0^{\frac{a}{2}}$$

$$= \frac{4}{3} \pi \left( a^3 - \frac{3\sqrt{3}}{8} a^3 \right) = \pi a^3 \left( \frac{4}{3} - \frac{\sqrt{3}}{2} \right)$$

P.65

5.2 (1)  $0 \leq z \leq 1 - (x^2 + y^2)^{\frac{1}{2}}$

$$V = \iint_D \{1 - (x^2 + y^2)^{\frac{1}{2}}\} dx dy$$

$$= \int_0^{2\pi} d\theta \int_0^1 (1 - r^{\frac{1}{2}}) r dr$$

$$= 2\pi \left[ \frac{r^2}{2} - \frac{2}{5} r^{\frac{5}{2}} \right]_0^1 = \frac{\pi}{5}$$

D:  $x^2 + y^2 \leq 1$   
 $x = r \cos \theta$   $y = r \sin \theta$   $0 \leq r \leq 1$   
 $J = r$

(2)  $x^2 + y^2 + z^2 \leq 2$ ,  $x^2 + y^2 \leq 1$

D:  $x^2 + y^2 \leq 1$

$$V = \iint_D 2\sqrt{2 - x^2 - y^2} dx dy$$

$$= 2 \int_0^{2\pi} d\theta \int_0^1 \sqrt{2 - r^2} r dr$$

$$= 2 \int_0^{2\pi} \left[ -\frac{1}{3} (2 - r^2)^{\frac{3}{2}} \right]_0^1 d\theta = 2 \int_0^{2\pi} \left( \frac{2\sqrt{2}}{3} - \frac{1}{3} \right) d\theta = \frac{4\pi}{3} (2\sqrt{2} - 1)$$

$x = r \cos \theta$   $y = r \sin \theta$   $0 \leq r \leq 1$   
 $J = r$

(3)  $x^2 + y^2 \leq 1$ ,  $z \geq -y$ ,  $z \leq 2y$

$-y \leq z \leq 2y$

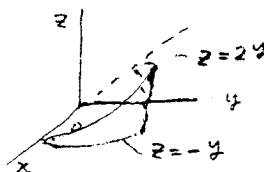
$$V = \int_0^\pi \int_0^\pi 3y dx dy$$

$$= 2 \int_0^\pi d\theta \int_0^1 3r^2 \sin \theta dr$$

$$= 6 \int_0^\pi \sin \theta d\theta \int_0^1 r^2 dr$$

$$= 6 \left[ -\cos \theta \right]_0^\pi \left[ \frac{r^3}{3} \right]_0^1 = 4$$

D:  $x^2 + y^2 \leq 1$   $x = r \cos \theta$   $0 \leq r \leq 1$   
 $y = r \sin \theta$   $J = r$



(4)  $x^2 + y^2 + z^2 \leq 1$ ,  $x + y + z = 1$

$$V = \int_{\frac{1}{\sqrt{3}}}^1 \pi(1 - x^2) dx$$

$$= \pi \left[ x - \frac{x^3}{3} \right]_{\frac{1}{\sqrt{3}}}^1 = \pi \left( \frac{2}{3} - \frac{1}{3\sqrt{3}} + \frac{1}{9\sqrt{3}} \right) = \pi \left( \frac{2}{3} - \frac{2}{9\sqrt{3}} \right) = \frac{2\pi}{9\sqrt{3}} (2\sqrt{3} - 1)$$

$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$



(5)  $x^2 + y^2 = a^2$ ,  $0 \leq z \leq x + y$

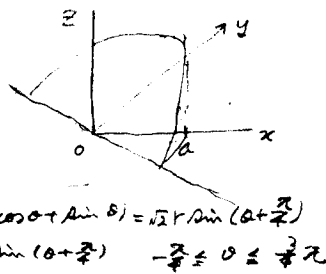
$$\iint_D (x + y) dx dy$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^a r^2 \sqrt{2} \sin(\theta + \frac{\pi}{4}) dr$$

$$= \sqrt{2} \left[ -\cos(\theta + \frac{\pi}{4}) \right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[ \frac{r^3}{3} \right]_0^a$$

$$= 2\sqrt{2} \frac{a^3}{3} = \frac{2\sqrt{2}}{3} a^3$$

D:  $x^2 + y^2 \leq a^2$ ,  $0 \leq x + y$   
 $x = r \cos \theta$ ,  $y = r \sin \theta$   
 $0 \leq r \leq a$ ,  $0 \leq r(\cos \theta + \sin \theta) = \sqrt{2} r \sin(\theta + \frac{\pi}{4})$   
 $0 \leq \sin(\theta + \frac{\pi}{4})$   $-\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$



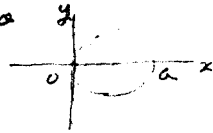
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(6)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$   $z = \pm c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

$V = \iint_D 2c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy$   $D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$   
 $= \int_0^{2\pi} d\theta \int_0^1 2abc \sqrt{1-r^2} r dr$   $x = ar \cos \theta$   $J = ab r$   
 $y = br \sin \theta$   
 $= 2abc \cdot 2\pi \cdot \left[ -\frac{1}{3} (1-r^2)^{\frac{3}{2}} \right]_0^1 = \frac{4}{3} abc \pi$

(7)  $x^2 + y^2 \leq a^2$   $x^2 + y^2 + z^2 = a^2$

$V = \iint_D 2\sqrt{a^2 - x^2 - y^2} dx dy$   $z = \pm \sqrt{a^2 - x^2 - y^2}$   $J = r$   
 $D: x^2 + y^2 \leq a^2$   $x = r \cos \theta$   $y = r \sin \theta$   $r \leq a \cos \theta$   
 $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{a \cos \theta} 2\sqrt{a^2 - r^2} r dr$   
 $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \left[ -\frac{1}{3} (a^2 - r^2)^{\frac{3}{2}} \right]_0^{a \cos \theta} d\theta$   
 $= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} (a^3 - a^3 \cos^3 \theta) d\theta = \frac{2a^3}{3} \cdot 2 \left( \frac{\pi}{2} - \frac{2}{3} \right) = \frac{2}{3} a^3 \left( \pi - \frac{4}{3} \right)$

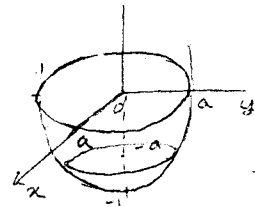


(8)  $x^2 + y^2 \leq 1$ ,  $0 \leq z \leq y$

$\iint_D y dx dy$   $D: x^2 + y^2 \leq 1, y \geq 0$   
 $x = r \cos \theta, y = r \sin \theta, J = r, 0 \leq \theta \leq \pi$   
 $= \int_0^{\pi} \int_0^1 r^3 \sin \theta dr d\theta$   
 $= \left[ \cos \theta \right]_0^{\pi} \left[ \frac{r^4}{4} \right]_0^1 = \frac{2}{3}$

5.3

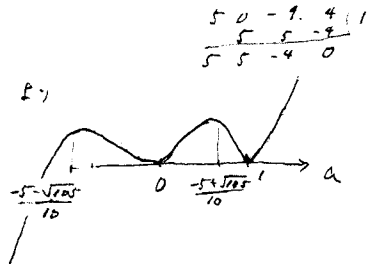
$\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 \leq 1$  ( $0 < a \leq 1$ )  $z = -a$   
 $z = -\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}}$



$V(a) = \iint_D (\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}} - a) dx dy$   $D: x^2 + y^2 \leq a^2(1 - a^2)$   
 $= \int_0^{2\pi} d\theta \int_0^{\sqrt{1-a^2}} (\sqrt{1-r^2} - a) a^2 r dr$   $x = ar \cos \theta, y = ar \sin \theta, J = a^2 r$   
 $r^2 \leq (1-a^2), r \leq \sqrt{1-a^2}$   
 $= 2\pi a^2 \left[ -\frac{1}{3} (1-r^2)^{\frac{3}{2}} - a \frac{r^2}{2} \right]_0^{\sqrt{1-a^2}}$   
 $= 2\pi a^2 \left( \frac{1}{3} - \frac{1}{3} a^3 - \frac{a}{2} + \frac{a^3}{2} \right) = \frac{\pi}{3} a^2 (a^3 - 3a + 2)$

$V'(a) = \frac{\pi}{3} (5a^4 - 9a^2 + 4a) = \frac{\pi}{3} a (5a^2 - 9a + 4)$   
 $= \frac{\pi}{3} a (a-1)(5a^2 + 5a - 4)$   $0 < a \leq 1$

$a = \frac{-5 \pm \sqrt{25}}{10}, a = \frac{-5 + \sqrt{25}}{10}$



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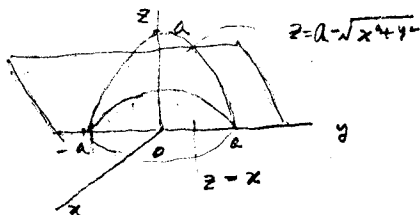
5.4

$$z = a - \sqrt{x^2 + y^2} \quad x=0, \quad z=x$$

$$x = a - \sqrt{x^2 + y^2} \quad x^2 + y^2 = (a-x)^2$$

$$y^2 = a^2 - 2ax$$

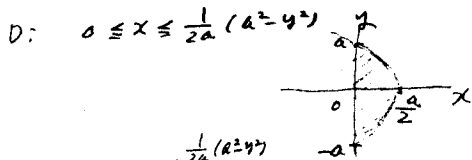
$$x = \frac{a}{2} - \frac{y^2}{2a}$$



$$V = \iint_D \{a - \sqrt{x^2 + y^2} - x\} dx dy$$

$$= \int_{-a}^a dy \int_0^{\frac{1}{2a}(a^2 - y^2)} \{a - x - \sqrt{x^2 + y^2}\} dx$$

$$= \int_{-a}^a \left[ -\frac{1}{2}(a-x)^2 - \frac{1}{2} \{x\sqrt{x^2 + y^2} + y^2 \log(x + \sqrt{x^2 + y^2})\} \right]_{x=0}^{\frac{1}{2a}(a^2 - y^2)} dy$$



$$= \int_{-a}^a \left[ \frac{1}{2}a^2 - \frac{1}{2} \left(\frac{a^2 - y^2}{2a}\right)^2 - \frac{1}{2} \frac{a^2 - y^2}{2a} \cdot \frac{a^2 - y^2}{2a} - \frac{y^2}{2} \log\left(\frac{a^2 - y^2}{2a} + \frac{a^2 - y^2}{2a}\right) + \frac{y^2}{2} \log|y| \right] dy$$

$$= \int_{-a}^a \left[ \frac{1}{2}a^2 - \frac{1}{2} \frac{a^2 + a^2 - 2ay^2}{2a^2} - \frac{y^2}{2} \log a + \frac{y^2}{2} \log|y| \right] dy$$

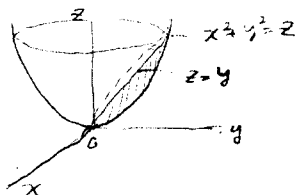
$$= 2 \left[ \frac{1}{2}a^2 y - \frac{a^2}{4} y - \frac{1}{4} \frac{y^3}{3} - \frac{y^3}{8} \log a + \frac{y^3}{8} \log|y| - \frac{y^3}{18} \right]_0^a$$

$$= 2 \left( \frac{1}{2}a^3 - \frac{a^3}{4} - \frac{a^3}{12} - \frac{a^3}{18} \right) = \frac{2}{9}a^3$$

5.5

$$x^2 + y^2 = z \quad z = y$$

$$x^2 + y^2 = y$$



$$\iint_D \{y - (x^2 + y^2)\} dx dy$$

$$D: x^2 + y^2 \leq y$$

$$= \int_0^\pi d\theta \int_0^{\sin \theta} (r^2 \sin \theta - r^3) dr$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r \leq \sin \theta \\ J = r \end{cases}$$

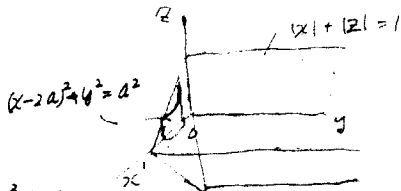
$$= \int_0^\pi \left( \frac{1}{3} \sin^3 \theta - \frac{1}{4} \sin^4 \theta \right) d\theta = \frac{1}{12} \int_0^\pi \sin^3 \theta d\theta = \frac{1}{12} \cdot 2 \cdot \frac{4}{3} = \frac{1}{3}$$

5.6

$$(x-2a)^2 + y^2 \leq a^2 \quad |z| + |\bar{z}| \leq 1$$

$$0 < 3a \leq 1$$

$$x-1 \leq z \leq 1-x$$



$$\iint_D \{(1-x) - (x-1)\} dx dy \quad D: (x-2a)^2 + y^2 \leq a^2$$

$$= \int_a^{3a} dx \int_{-\sqrt{a^2 - (x-2a)^2}}^{\sqrt{a^2 - (x-2a)^2}} 2(1-x) dy = 4 \int_a^{3a} (1-x) \sqrt{a^2 - (x-2a)^2} dx$$

$$y = \pm \sqrt{a^2 - (x-2a)^2}$$

$$x-2a = a \cos \theta$$

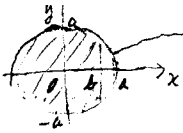
$$dx = -a \sin \theta d\theta$$

$$= 4 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1-2a-a \cos \theta) a^2 \cos^2 \theta d\theta = 4a^2(1-2a) \frac{\pi}{2} + 4a^3 \left[ \frac{2}{3} \cos^3 \theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= 2a^2 \pi (1-2a)$$

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5.7



$$y^2 + z^2 = a^2 \quad y^2 = a^2 - z^2$$

$$V(b) = \pi \int_{-a}^b (a^2 - z^2) dz = \pi \left[ a^2 z - \frac{1}{3} z^3 \right]_{-a}^b$$

$$= \pi \left( \frac{2}{3} a^3 + a^2 b - \frac{1}{3} b^3 \right)$$

$$S(b) = 2\pi \int_{-a}^b y \sqrt{1+y'^2} dz = 2\pi \int_{-a}^b \sqrt{a^2 - z^2} \sqrt{1 + \frac{z^2}{a^2 - z^2}} dz = 2\pi a \int_{-a}^b \frac{dz}{\sqrt{a^2 - z^2}}$$

$$= 2\pi a (b + a)$$

$$\therefore \frac{V(b)}{S(b)} = \frac{1}{2a(b+a)} \left( \frac{2}{3} a^3 + a^2 b - \frac{1}{3} b^3 \right) = \frac{1}{2} \frac{1}{a+b} \left( \frac{2}{3} a^3 + a^2 b - \frac{1}{3} b^3 \right)$$

$$F(b) = V(b)/S(b) \quad 0 < b < 2a$$

$$F'(b) = \frac{1}{2} \left\{ \frac{1}{a+b} (a^2 - b^2) - \frac{a}{(a+b)^2} \left( \frac{2}{3} a^3 + a^2 b - \frac{1}{3} b^3 \right) \right\}$$

$$= \frac{1}{2(a+b)^2} \left\{ a^3 + a^2 b - a^2 b^2 - ab^2 - \frac{2}{3} a^4 - a^3 b + \frac{1}{3} ab^3 \right\}$$

$$= \frac{1}{2(a+b)^2} \left( \frac{1}{3} a^3 - a^2 b^2 - \frac{2}{3} ab^3 \right) = \frac{1}{6a(a+b)^2} (a^3 - 3ab^2 - 2b^3)$$

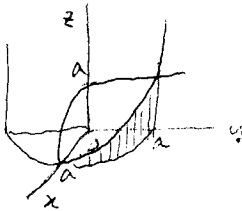
$$= \frac{1}{6a(a+b)^2} (a+b)^2 (a-2b)$$

$$= \frac{1}{6a} (a-2b) \quad F'(b) = 0 \Leftrightarrow b = \frac{1}{2} a$$

$$\begin{array}{cccc} 1 & 0 & -3b^2 & -2b^3 & 1 & -b \\ & -b & b^2 & 2b^3 & & \\ & 1 & -b & -2b^2 & 0 & \\ & & -b & 2b^2 & & \\ & & 1 & -2b & 0 & \end{array}$$

$$\therefore h = a + b = \frac{3}{2} a //$$

5.8



$$x^2 + y^2 = a^2, \quad x^2 + y^2 = a^2$$

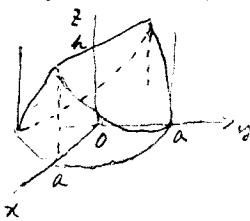
$$V = 8 \iint_D \sqrt{a^2 - x^2} dx dy$$

$$D: x^2 + y^2 \leq a^2, \quad x \geq 0$$

$$= 8 \int_0^a dx \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2} dy = 8 \int_0^a (a^2 - x^2) dx$$

$$= 8 \left[ a^2 x - \frac{x^3}{3} \right]_0^a = \frac{16}{3} a^3$$

5.9 尚是更难的图在记入



$$x^2 + y^2 = a^2, \quad z = -\frac{h}{a} y + h$$

$$V = 2 \iint_D \left( -\frac{h}{a} y + h \right) dx dy$$

$$D: x^2 + y^2 \leq a^2, \quad y \geq 0$$

$$= 2 \int_0^a dy \int_0^{\sqrt{a^2 - y^2}} \left( -\frac{h}{a} r \sin \theta + h r \right) dr$$

$$x = r \cos \theta$$

$$J = r$$

$$y = r \sin \theta$$

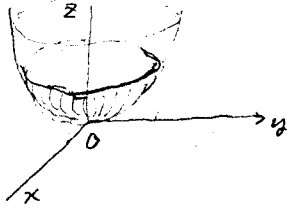
$$0 \leq r \leq a$$

$$0 \leq \theta \leq \pi$$

$$= 2 \int_0^a \left( -\frac{a^2 h}{3} \sin \theta + \frac{h a^2}{2} \right) d\theta = 2 a^2 h \left[ -\frac{1}{3} \cos \theta + \frac{\theta}{2} \right]_0^\pi$$

$$= 2 a^2 h \left( -\frac{2}{3} + \frac{\pi}{2} \right)$$

5.10  $x^2 + y^2 = z$       $2x^2 + y^2 = a^2$



$$z = x^2 + y^2, \quad \frac{x^2}{\frac{a^2}{2}} + \frac{y^2}{a^2} = 1$$

$$y=0, \quad x = \pm \frac{a}{\sqrt{2}}$$

$$\int \int_D (x^2 + y^2) dx dy$$

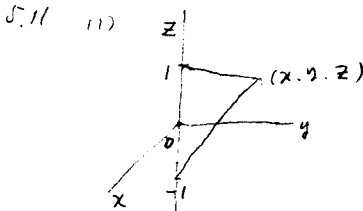
$$= \int_0^{2\pi} d\theta \int_0^{\frac{a}{\sqrt{2}}} r^2 dr = \frac{a^2 \cdot 2\pi}{4 \cdot 4}$$

$$= \frac{a^2 \pi}{8}$$

$D: x^2 + y^2 \leq \frac{a^2}{2}$

$x = r \cos \theta$       $J = r$   
 $y = r \sin \theta$       $0 \leq r \leq \frac{a}{\sqrt{2}}$

求める体積  $\frac{a^2}{2} \pi \cdot \frac{a^2}{2} - \frac{a^4}{8} \pi = \frac{a^4}{8} \pi$



$$\sqrt{x^2 + y^2 + (z+1)^2} + \sqrt{x^2 + y^2 + (z-1)^2} = 2a$$

$$\frac{\sqrt{x^2 + y^2 + (z+1)^2} - \sqrt{x^2 + y^2 + (z-1)^2}}{4z} = \frac{1}{2a}$$

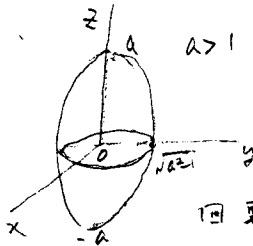
$$\sqrt{x^2 + y^2 + (z+1)^2} - \sqrt{x^2 + y^2 + (z-1)^2} = 2 \frac{z}{a}$$

$$\sqrt{x^2 + y^2 + (z+1)^2} = a + \frac{z}{a}$$

$$x^2 + y^2 + z^2 + 2z + 1 = \frac{z^2}{a^2} + 2z + a^2$$

$$x^2 + y^2 + \frac{a^2-1}{a^2} z^2 = a^2 - 1$$

$$\frac{x^2}{a^2-1} + \frac{y^2}{a^2-1} + \frac{z^2}{a^2} = 1$$



回転体

$0 < a < 1$  のとき条件に適合する点 P は z 軸上の  $-1 \leq z \leq 1$  のみ

(2)

$$\int \int_D 2a \sqrt{1 - \frac{x^2}{a^2-1} - \frac{y^2}{a^2-1}} dx dy$$

$$= 2a \int_0^{2\pi} d\theta \int_0^1 \sqrt{1-r^2} (a^2-1) r dr$$

$$= 2a(a^2-1) 2\pi \left[ -\frac{1}{3} (1-r^2)^{\frac{3}{2}} \right]_0^1$$

$$= \frac{4\pi a}{3} (a^2-1)$$

$D: x^2 + y^2 \leq a^2 - 1$

$x = r \sqrt{a^2-1} \cos \theta$   
 $y = r \sqrt{a^2-1} \sin \theta$   
 $J = (a^2-1) r$



P.66

## § 6 重積分の応用 (面積)

$$6.1 (1) \quad z = \tan^{-1} \frac{y}{x} \quad x > 0, y > 0, \quad x^2 + y^2 \leq 1$$

$$z_x = \frac{-y}{x^2 + y^2} \quad z_y = \frac{x}{x^2 + y^2} \quad 1 + z_x^2 + z_y^2 = 1 + \frac{1}{x^2 + y^2} = \frac{1 + x^2 + y^2}{x^2 + y^2}$$

$$\begin{aligned} & \iint_D \sqrt{\frac{1 + x^2 + y^2}{x^2 + y^2}} dx dy \quad D: x^2 + y^2 \leq 1, x > 0, y > 0 \\ & = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \frac{\sqrt{1+r^2}}{r} r dr \quad \begin{array}{l} x = r \cos \theta \quad J = r \quad 0 \leq r \leq 1 \\ y = r \sin \theta \quad \quad \quad \quad \quad \quad \quad 0 \leq \theta \leq \frac{\pi}{2} \end{array} \\ & = \int_0^{\frac{\pi}{2}} \frac{1}{2} \left[ r \sqrt{1+r^2} + \log |r + \sqrt{r^2 + 1}| \right]_0^1 d\theta = \frac{1}{2} (\sqrt{2} + \log(1 + \sqrt{2})) \int_0^{\frac{\pi}{2}} d\theta \\ & = \frac{\pi}{4} (\sqrt{2} + \log(1 + \sqrt{2})) \end{aligned}$$

$$(2) \quad x^2 + y^2 + z^2 = a^2, \quad z \geq b, \quad 0 \leq b \leq a$$

$$z = \sqrt{a^2 - x^2 - y^2} \quad z_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \quad z_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$1 + z_x^2 + z_y^2 = \frac{a^2}{a^2 - x^2 - y^2}$$

$$\begin{aligned} & \iint_D \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy \quad \begin{array}{l} a^2 - x^2 - y^2 \geq b^2 \quad x = r \cos \theta \\ a^2 - b^2 \geq x^2 + y^2 \quad y = r \sin \theta \\ 0 \leq r \leq \sqrt{a^2 - b^2}, \quad 0 \leq \theta \leq 2\pi \end{array} \\ & = \int_0^{2\pi} d\theta \int_0^{\sqrt{a^2 - b^2}} \frac{a}{\sqrt{a^2 - r^2}} r dr \\ & = \int_0^{2\pi} \left[ -a \sqrt{a^2 - r^2} \right]_0^{\sqrt{a^2 - b^2}} d\theta = \int_0^{2\pi} (a^2 - ab) d\theta = 2\pi a(a - b) \end{aligned}$$

$$13. \quad \Gamma - \Gamma, \text{ 面} \quad x = (a + b \sin \theta) \cos \phi \quad y = (a + b \sin \theta) \sin \phi \quad z = b \cos \theta$$

$$a > b > 0$$

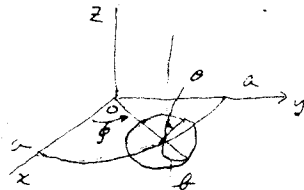
$$z^2 + (x - a)^2 = b^2$$

$$x = a \pm \sqrt{b^2 - z^2}$$

$$\frac{dx}{dz} = \mp \frac{z}{\sqrt{b^2 - z^2}}$$

$$1 + \left(\frac{dx}{dz}\right)^2 = \frac{b^2}{b^2 - z^2}$$

$$\begin{aligned} & 2\pi \int_{-b}^b \frac{b}{(a + \sqrt{b^2 - z^2}) \sqrt{b^2 - z^2}} dz + 2\pi \int_{-b}^b \frac{b}{(a - \sqrt{b^2 - z^2}) \sqrt{b^2 - z^2}} dz \\ & = 4\pi ab \int_{-b}^b \frac{1}{\sqrt{b^2 - z^2}} dz = 4\pi ab \left[ \sin^{-1} \frac{z}{b} \right]_{-b}^b = 4\pi^2 ab \end{aligned}$$



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6.1 (4)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a, b, c > 0)$

$z = c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad z_x = c \left( \frac{-x}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \right) \frac{1}{a^2} \quad z_y = c \left( \frac{-y}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \right) \frac{1}{b^2}$

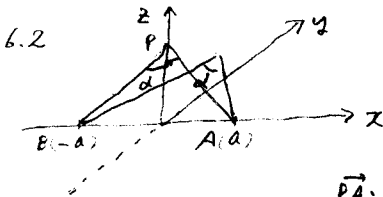
$1 + z_x^2 + z_y^2 = 1 + \frac{c^2}{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)} \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right)$   
 $= \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + c^2 \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right) \right) / \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)$

$\iint_D \left\{ 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + c^2 \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right) \right\}^{1/2} / \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{1/2} dx dy$

$= \int_0^{2\pi} d\theta \int_0^1 \frac{\sqrt{1-r^2 + c^2 \left( \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)}}{\sqrt{1-r^2}} r dr$   
 $x = ar \cos \theta \quad y = br \sin \theta \quad J = ab r$

$= \int_0^{2\pi} d\theta \int_0^1 \sqrt{1 + c^2 \left( \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)} \frac{r^2}{1-r^2} dr$

∴ 求 a, b, c z について

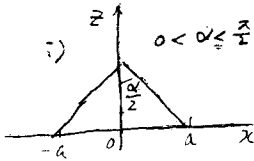


$P(x, y, z)$  とおく

$\vec{PA} = (a-x, -y, -z)$

$\vec{PB} = (-a-x, -y, -z)$

$\cos \alpha = \frac{\vec{PA} \cdot \vec{PB}}{|\vec{PA}| |\vec{PB}|} = \frac{x^2 - a^2 + y^2 + z^2}{\sqrt{(a-x)^2 + y^2 + z^2} \sqrt{(a+x)^2 + y^2 + z^2}}$



$0 < \alpha \leq \frac{\pi}{2} \quad a \leq z$

$y = \frac{1}{2} \left( \frac{a}{\tan \frac{\alpha}{2}} + a \tan \frac{\alpha}{2} \right), \quad b = \frac{1}{2} \left( \frac{a}{\tan \frac{\alpha}{2}} - a \tan \frac{\alpha}{2} \right)$

$z = b + \sqrt{r^2 - x^2} \quad \frac{dz}{dx} = \frac{-x}{\sqrt{r^2 - x^2}} \quad 1 + \left( \frac{dz}{dx} \right)^2 = \frac{r^2}{r^2 - x^2}$

$\frac{S}{2} = 2\pi \int_0^r (b + \sqrt{r^2 - x^2}) \frac{r}{\sqrt{r^2 - x^2}} dx + 2\pi \int_a^r (b + \sqrt{r^2 - x^2}) \frac{r}{\sqrt{r^2 - x^2}} dx$

$= 2\pi \int_0^r \left( \frac{br}{\sqrt{r^2 - x^2}} + r \right) dx + 2\pi \int_a^r \left( \frac{br}{\sqrt{r^2 - x^2}} + r \right) dx$

$= 2\pi \left[ br \sin^{-1} \frac{x}{r} + rx \right]_0^r + 2\pi \left[ br \sin^{-1} \frac{x}{r} + rx \right]_a^r$

$= 2\pi \left( \frac{br\pi}{2} + r^2 + \frac{br\pi}{2} + r^2 - br \sin^{-1} \frac{a}{r} - ra \right)$

$= 2\pi r (b\pi + 2r - b \sin^{-1} \frac{a}{r} - a)$

$\therefore S = 4\pi r (b\pi + 2r - b \sin^{-1} \frac{a}{r} - a)$

ii)  $\frac{\pi}{2} < \alpha < \pi \quad a < z$

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$$r = \frac{a}{2} \left( \frac{1}{\tan \frac{\alpha}{2}} + \tan \frac{\alpha}{2} \right), \quad b = \frac{a}{2} \left( \frac{1}{\tan \frac{\beta}{2}} - \tan \frac{\beta}{2} \right)$$

$$(z+b)^2 + x^2 = r^2 \quad z = -b + \sqrt{r^2 - x^2}$$

$$S = 4\pi \int_0^a (-b + \sqrt{r^2 - x^2}) \frac{r}{\sqrt{r^2 - x^2}} dx = 4\pi \int_0^a \left( \frac{-br}{\sqrt{r^2 - x^2}} + r \right) dx$$

$$= 4\pi \left[ -br \sin^{-1} \frac{x}{r} + rx \right]_0^a = 4\pi (ra - br \sin^{-1} \frac{a}{r})$$

6.3 (1)  $x = a u \cos v, y = b u \sin v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a \cos v & -au \sin v \\ b \sin v & bu \cos v \end{vmatrix} = ab u$$

(2)  $\iint_D dx dy \quad D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$

$$= \int_0^{2\pi} d\theta \int_0^1 ab u du = 2\pi \frac{ab}{2} = ab\pi$$

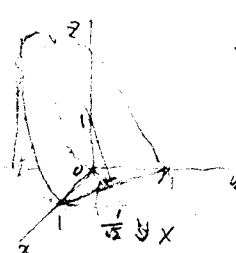
6.4 6.10 (2)  $|\vec{a}| |\vec{b}|$

6.5  $x^2 + y^2 = 1 \quad x + y + z \leq 1 \quad \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$

$$0 \leq z \leq 1 - x - y$$

$$z = -\sqrt{2}x + 1$$

$$\begin{cases} x = x \cos \frac{\pi}{4} - y \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}(x - y) \\ y = x \sin \frac{\pi}{4} + y \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}(x + y) \end{cases} \text{ 旋转 } \pi/4$$



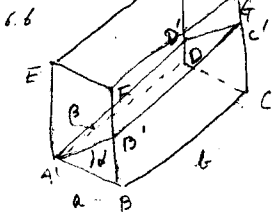
$$x^2 + y^2 = 1$$

$$y = \sqrt{1-x^2} \quad \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$1 + y'^2 = \frac{1}{1-x^2}$$

$$S = 2 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (-\sqrt{2}x + 1) \frac{1}{\sqrt{1-x^2}} dx = 2 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left( \sqrt{2} \frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= 2 \left[ \sqrt{2} \sqrt{1-x^2} + \sin^{-1} x \right]_{-1/\sqrt{2}}^{1/\sqrt{2}} = 2 \left( 1 + \frac{\pi}{4} + \frac{\pi}{2} \right) = 2 + \frac{3}{2}\pi$$



$A(0,0,0) \quad B'(a,0,a \tan \alpha) \quad C'(0,b,b \tan \beta)$

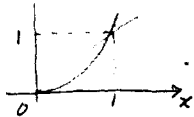
$$\vec{AB}' \times \vec{AC}' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & 0 & a \tan \alpha \\ 0 & b & b \tan \beta \end{vmatrix} = \vec{i} (ab \tan \alpha + a^2 \tan \alpha \tan \beta) + \vec{j} (ab \tan \beta) + \vec{k} ab$$

$$S = |\vec{AB}' \times \vec{AC}'| = \sqrt{a^2 b^2 \tan^2 \alpha + a^2 b^2 \tan^2 \beta + a^2 b^2}$$

$$= ab \sqrt{1 + \tan^2 \alpha + \tan^2 \beta}$$

## P. 46 § 7 重積分の応用 (物理)

7.1  $y=x, y=x^2$



$$S = \int_0^1 (x-x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$\int_0^1 x(x-x^2) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}$$

$$\int_0^1 y(\sqrt{y}-y) dy = \left[ \frac{2}{5} y^{5/2} - \frac{y^3}{3} \right]_0^1 = \frac{1}{15}$$

$$\bar{x} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2} \quad \bar{y} = \frac{\frac{1}{15}}{\frac{1}{6}} = \frac{2}{5} \quad \text{重心} \left( \frac{1}{2}, \frac{2}{5} \right)$$

7.2  $z = \sqrt{a^2 - (x^2 + y^2)} \quad \bar{z}_x = \frac{-x}{\sqrt{a^2 - (x^2 + y^2)}} \quad \bar{z}_y = \frac{-y}{\sqrt{a^2 - (x^2 + y^2)}}$

$$S = \iint_D \sqrt{1+x^2+y^2} dx dy = \iint_D \frac{a}{\sqrt{a^2 - (x^2 + y^2)}} dx dy \quad D: x^2 + y^2 \leq a^2$$

$$= \int_0^{2\pi} d\theta \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r dr = 2\pi a \left[ -\sqrt{a^2 - r^2} \right]_0^a = 2\pi a^2 \quad \begin{matrix} x = r \cos \theta & r = r \\ y = r \sin \theta & 0 \leq r \leq a \end{matrix}$$

$$\iint_D \sqrt{a^2 - (x^2 + y^2)} \frac{a}{\sqrt{a^2 - (x^2 + y^2)}} dx dy = \int_0^{2\pi} d\theta \int_0^a a r dr = 2\pi \frac{a^3}{2} = \pi a^3$$

$$\bar{z} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2} \quad \text{重心} \left( 0, 0, \frac{a}{2} \right)$$

7.3  $x^2 + y^2 + z^2 \leq a^2, z \geq 0$

$$V = \int_0^{2\pi} \int_0^a \sqrt{a^2 - (x^2 + y^2)} dx dy = \int_0^{2\pi} d\theta \int_0^a \sqrt{a^2 - r^2} r dr \quad D: x^2 + y^2 \leq a^2$$

$$= 2\pi \left[ -\frac{1}{3} (a^2 - r^2)^{3/2} \right]_0^a = \frac{2}{3} \pi a^3 \quad \begin{matrix} x = r \cos \theta & r = r \\ y = r \sin \theta & 0 \leq r \leq a \end{matrix}$$

$$\iiint_{D_1} z dx dy dz \quad D_1: x^2 + y^2 + z^2 \leq a^2$$

$$= \int_0^{2\pi} \int_0^a \frac{z^2}{2} dx dy = \frac{1}{2} \iint_D (a^2 - (x^2 + y^2)) dz dx dy$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^a (a^2 - r^2) r dr = \frac{1}{2} \cdot 2\pi \left[ \frac{a^2}{2} r^2 - \frac{r^4}{4} \right]_0^a = \frac{\pi a^4}{4}$$

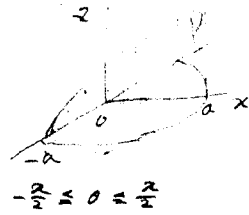
$$\bar{z} = \frac{\frac{1}{4} \pi a^4}{\frac{2}{3} \pi a^3} = \frac{3}{8} a$$

$$\text{重心} \left( 0, 0, \frac{3}{8} a \right)$$

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§ 8. 総合問題

8.1 (1)  $x^2 + y^2 \leq a^2$  (円)  $z \leq kx, z \geq 0$



(2)  $\int_D kx \, dx \, dy$       $D: x^2 + y^2 \leq a^2, x \geq 0$   
 $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^a r^2 k \cos \theta \, dr$       $x = r \cos \theta$   
 $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{k^3}{3} \cos \theta \, d\theta = \frac{k^3}{3} [\sin \theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2k^3}{3}$       $y = r \sin \theta$       $J = r$       $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(3) 底面積  $\frac{1}{2} \pi a^2$       $y = \pm \sqrt{a^2 - x^2}$       $1 + y'^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}$   
 側面積  $2 \int_0^a kx \frac{a}{\sqrt{a^2 - x^2}} dx = 2ak [-\sqrt{a^2 - x^2}]_0^a = 2a^2k$   
 $z = kx$       $z_x = k$       $z_y = 0$       $1 + z_x^2 + z_y^2 = 1 + k^2$   
 上面積  $\iint_D \sqrt{1+k^2} \, dx \, dy = \frac{\pi}{2} a^2 \sqrt{1+k^2}$       $D: x^2 + y^2 \leq a^2, x \geq 0$   
 $\therefore \frac{\pi a^2}{2} (1 + \sqrt{1+k^2}) + 2a^2k$

(4) (2)より  $\frac{2a^3k}{3} = V_0$  (-定)      $k^3 = \frac{3V_0}{2a^3}$

$\therefore S = \frac{\pi}{2} a^2 \sqrt{1+k^2}$

$\pi a \sqrt{1+k^2} + \lambda 3a^2k = 0$       $\frac{2}{9} \pi a^2 \sqrt{1+k^2} + \lambda V_0 = 0$

$\frac{2}{3} a^2 \frac{k}{\sqrt{1+k^2}} + \lambda a^3 = 0$       $\frac{\pi}{3} k^2 \frac{1}{\sqrt{1+k^2}} + \lambda V_0 = 0$

$\frac{2}{3} \pi a^2 \sqrt{1+k^2} = \pi a^2 k^2 \frac{1}{\sqrt{1+k^2}}$       $2(1+k^2) = 3k^2$

$k^2 = 2 \quad \therefore k = \sqrt{2} \quad a^3 = \frac{3}{2\sqrt{2}} V_0 //$

8.2 (1)  $f(x, y) = \int_{x-2y}^{x+2y} g(t) \, dt$       $G(x) = \int g(t) \, dt$       $G(x) = G(x)$

$f(x, y) = [G(t)]_{x-2y}^{x+2y} = G(x+2y) - G(x-2y)$

$f_x(x, y) = g(x+2y) - g(x-2y)$

$f_y(x, y) = 2g(x+2y) - 2g(x-2y)$

$f_x(x, 0) = 0$       $f_y(x, 0) = 4g(x)$

(2)  $f_{xx}(x, y) = g'(x+2y) - g'(x-2y)$

$f_{yy}(x, y) = 4(g'(x+2y) - g'(x-2y))$

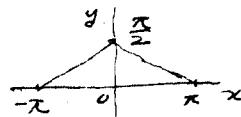
$\therefore f_{yy}(x, y) = C f_{xx}(x, y) \quad C = 4$

$\frac{d}{dx} g(x) = g'(x) \quad x < c$

8.2 (3)  $g(x) = \cos x \quad x \in \mathbb{R}$

$$f(x, y) = \int_{x-2y}^{x+2y} \cos t \, dt = \left[ \sin t \right]_{x-2y}^{x+2y} = \sin(x+2y) - \sin(x-2y)$$

$$= 2 \cos x \sin 2y \quad D: y \geq 0$$



$$\iint_D f(x, y) \, dx \, dy$$

$$= \int_0^{\pi/2} dy \int_{2y-\pi}^{2y+\pi} 2 \cos x \sin 2y \, dx = 2 \int_0^{\pi/2} \sin 2y \left[ \sin x \right]_{2y-\pi}^{2y+\pi} dy$$

$$= 2 \int_0^{\pi/2} \sin 2y \{ \sin(-2y+\pi) - \sin(2y-\pi) \} dy$$

$$= 4 \int_0^{\pi/2} \sin 2y \sin(-2y+\pi) dy = 4 \int_0^{\pi/2} \sin^2 2y dy \quad \begin{matrix} 2y = t \\ dy = \frac{1}{2} dt \end{matrix}$$

$$= 2 \int_0^{\pi} \sin^2 t \, dt = 2 \cdot 2 \cdot \frac{\pi}{4} = \pi$$

8.3 (1)  $x = \frac{r}{a} \cos \theta, \quad y = \frac{r}{b} \sin \theta \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{1}{a} \cos \theta & -\frac{r}{a} \sin \theta \\ \frac{1}{b} \sin \theta & \frac{r}{b} \cos \theta \end{vmatrix} = \frac{r}{ab}$

(2)  $\int_0^{\infty} f(x) \, dx = k, \quad G = \iint_D f(a^2x^2 + b^2y^2) \, dx \, dy \quad D: x \geq 0, y \geq 0$

$$G = \int_0^{\pi/2} d\theta \int_0^{\infty} f(r^2) \frac{r}{ab} dr \quad \begin{matrix} x = \frac{r}{a} \cos \theta, \quad y = \frac{r}{b} \sin \theta \\ r \geq t \end{matrix}$$

$$= \frac{1}{ab} \int_0^{\pi/2} d\theta \int_0^{\infty} \frac{1}{2} f(t) dt = \frac{k}{2ab} \cdot \frac{\pi}{2} = \frac{\pi k}{4ab}$$

(3)  $G = \frac{\pi k}{4ab} \quad 2a^2 + b^2 = 1 \quad \sqrt{2a^2 + b^2} \leq \frac{2a^2 + b^2}{2} = \frac{1}{2}$

$$|ab| \leq \frac{1}{2\sqrt{2}} \Rightarrow \frac{1}{|ab|} \geq 2\sqrt{2}$$

$$\therefore G \leq \frac{\pi k}{4} \cdot 2\sqrt{2} = \frac{\pi k}{\sqrt{2}}$$

最小値は  $2a^2 = b^2$  のとき  $\sqrt{2}a = \pm b$  のとき