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第8章 関数方程式

§.1 1階微分方程式

変数分離

1.1 (1) $y' = \dfrac{y}{x(x+1)(x+2)}$ $\quad y(1)=1$ $\quad \dfrac{1}{x(x+1)(x+2)} = \dfrac{1}{2}\left(\dfrac{1}{x(x+1)} - \dfrac{1}{(x+1)(x+2)}\right)$

$\dfrac{y'}{y} = \dfrac{1}{2x} - \dfrac{1}{x+1} + \dfrac{1}{2(x+2)}$ $\quad = \dfrac{1}{2}\left\{\dfrac{1}{x} - \dfrac{1}{x+1} - \left(\dfrac{1}{x+1} - \dfrac{1}{x+2}\right)\right\}$

$\therefore \log|y| = \dfrac{1}{2}\log\left|\dfrac{x(x+2)}{(x+1)^2}\right|$ $\quad = \dfrac{1}{2x} - \dfrac{1}{x+1} + \dfrac{1}{2(x+2)}$

$y = C\dfrac{\sqrt{x(x+2)}}{x+1}$ $\quad y(1)=1$ より $C\dfrac{\sqrt{3}}{2} = 1$ $\quad C = \dfrac{2}{\sqrt{3}}$

$\therefore y = \dfrac{2}{\sqrt{3}}\dfrac{\sqrt{x(x+2)}}{x+1}$ $\quad 3y^2(x+1)^2 = 4x(x+2)$

(2) $y' = 2\sqrt{y}$, $\quad y(0)=1$

$\dfrac{y'}{2\sqrt{y}} = 1$ $\quad \sqrt{y} = x + C$ $\quad y(0)=1$ より $C=1$ $\quad \therefore y = (x+1)^2$

(3) $y' = y\cos x$ $\quad \dfrac{y'}{y} = \cos x$ $\quad \log|y| = \sin x + C$

$\therefore y = a e^{\sin x}$

(4) $e^{x+y} + e^{2x-y}y' = 0$ $\quad e^x e^y + \dfrac{e^{2x}}{e^y}y' = 0$ $\quad e^{-x} + e^{-2y}y' = 0$

$-e^{-x} - \dfrac{1}{2}e^{-2y} = C$ $\quad 2e^{-x} + e^{-2y} = C$

1.2 (1) $y' = \dfrac{y^2 - x^2}{2xy}$ $\quad y = ux$ とおくと $y' = u'x + u$ $\quad u'x + u = \dfrac{u^2-1}{2u}$ $\quad$ 同次形

$u'x = -\dfrac{u^2+1}{2u}$ $\quad \dfrac{2u}{u^2+1}u' = -\dfrac{1}{x}$ $\quad \log|u^2+1| = -\log|x| + C$

$\therefore x(u^2+1) = a$ $\quad y^2 + x^2 = ax$

(2) $y' = \dfrac{2y - x}{x}$ $\quad y = ux$ とおくと $u'x + u = 2u - 1$, $u'x = u - 1$

$\dfrac{u'}{u-1} = \dfrac{1}{x}$ $\quad \log|u-1| = \log|x| + C$ $\quad u - 1 = ax$

$ux - x = ax^2$ $\quad y - x = ax^2$ $\quad \therefore y = x + ax^2$

(3) $xy' + x = ky$ $\quad y = ux$ とおくと $x(u'x + u) + x = kux$

$u'x + u + 1 = ku$ $\quad u'x = (k-1)u - 1$ $\quad \dfrac{u'}{(k-1)u-1} = \dfrac{1}{x}$ $\quad (k \neq 1)$

$\dfrac{1}{k-1}\log|(k-1)u-1| = \log|x| + C$ $\quad (k-1)u - 1 = ax^{k-1}$

$(k-1)ux - x = ax^k$ $\quad (k-1)y - x = ax^k$ $\quad y = \dfrac{x}{k-1} + bx^k$ $\quad (k \neq 1)$

$k=1$ のとき $u'x = -1$ $\quad u = -\log|x| + C$ $\quad y = x(-\log|x| + C)$

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1.2 (5) (5)→(4) $(x^2+y^2)dx = xy\,dy$ $\quad \frac{dy}{dx} = \frac{x^2+y^2}{xy}$

$y=ux$ とおくと $\quad u'x+u = \frac{1+u^2}{u} \quad u'x = \frac{1}{u}$, $uu' = \frac{1}{x}$

$\frac{1}{2}u^2 = \log|x| + C \quad u^2 = 2\log|x| + a \quad y^2 = x^2(\log x^2 + a)$

1.3 (1) $y'-2y=e^x$ 線形

$y = e^{2x}\{\int e^x e^{-2x}dx + C\} = e^{2x}(-e^{-x}+C) = Ce^{2x} - e^x$

(2) $y'+\frac{1}{x} y = \log x \quad \int \frac{1}{x} dx = \log x$

$y = \frac{1}{x}\{\int x\log x\,dx + C\} = \frac{1}{x}\{(\frac{x^2}{2}\log x - \frac{x^2}{4}) + C\}$

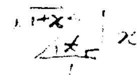
$= \frac{x}{2}\log x - \frac{x}{4} + \frac{C}{x}$

(3) $y' - y\tan x = \sin x \quad \int \tan x\,dx = \int\frac{\sin x}{\cos x}dx = -\log|\cos x|$

$y = \frac{1}{\cos x}\{\int \cos x\sin x\,dx + C\} = \frac{1}{\cos x}\{-\frac{1}{2}\cos^2 x + C\} = -\frac{1}{2}\cos x + \frac{C}{\cos x}$

(4) $(1+x^2)y' = xy+1 \quad y' - \frac{x}{1+x^2}y = \frac{1}{1+x^2} \quad \int\frac{x}{1+x^2}dx = \frac{1}{2}\log(1+x^2)$

$y = \sqrt{1+x^2}\{\int \frac{1}{\sqrt{1+x^2}}\frac{1}{1+x^2}dx + C\} \quad x=\tan t, \ dx = \sec^2 t\,dt$



$\int \frac{dx}{(\sqrt{1+x^2})^3} = \int\frac{1}{\sec^3 t}\sec^2 t\,dt = \int\cos t\,dt = \sin t$

$= \frac{x}{\sqrt{1+x^2}}$

$\therefore y = \sqrt{1+x^2}\{\frac{x}{\sqrt{1+x^2}} + C\} = x + C\sqrt{1+x^2}$

(5) $y'+xy = x \quad \int x\,dx = \frac{x^2}{2}$

$y = e^{-\frac{x^2}{2}}\{\int x e^{\frac{x^2}{2}}dx + C\} = e^{-\frac{x^2}{2}}(e^{\frac{x^2}{2}} + C)$

$= 1 + Ce^{-\frac{x^2}{2}}$

(6) $y' - \frac{1}{x} y = x^2$; $y(1) = \frac{3}{2}$, $\int\frac{1}{x} dx = \log x$

$y = x\{\int x\,dx + C\} = \frac{x^3}{2} + Cx$, $\frac{3}{2} = \frac{1}{2} + C \quad \therefore C=1$

$y = \frac{1}{2} x^3 + x$

(7) $y'+3y = x^2+1 \quad y = e^{-3x}\{\int(x^2+1)e^{3x}dx + C\}$

$= e^{-3x}\{\frac{x^2+1}{3}e^{3x} - \int\frac{2x}{3}e^{3x}dx + C\} = e^{-3x}\{\frac{3x^2+3-2x}{9}e^{3x} + \frac{2}{27}e^{3x} + C\}$

$= \frac{1}{27}(9x^2-6x+11) + Ce^{-3x}$

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1.4 (1) $(x+y)\,dx = y\,dy \qquad \dfrac{dy}{dx} = \dfrac{x+y}{y} \qquad y = ux$ とおくと

$u'x + u = \dfrac{1+u}{u} \qquad u'x = \dfrac{1+u-u^2}{u}$

$\dfrac{u}{u^2-u-1}\,u' = \dfrac{-1}{x} \qquad u^2-u-1=0 \quad u=\frac{1}{2}(1\pm\sqrt{5}) \qquad \dfrac{u}{u^2-u-1} = \dfrac{-1}{\sqrt{5}}\left(\dfrac{\frac{1-\sqrt{5}}{2}}{u-\frac{1-\sqrt{5}}{2}} - \dfrac{\frac{1+\sqrt{5}}{2}}{u-\frac{1+\sqrt{5}}{2}}\right)$

$\dfrac{1}{\sqrt{5}}\left\{\dfrac{\frac{1+\sqrt{5}}{2}}{u-\frac{1+\sqrt{5}}{2}} - \dfrac{\frac{1-\sqrt{5}}{2}}{u-\frac{1-\sqrt{5}}{2}}\right\}du = -\dfrac{1}{x}\,dx$

$\dfrac{1}{\sqrt{5}}\left\{\dfrac{1+\sqrt{5}}{2}\log\left|u-\dfrac{1+\sqrt{5}}{2}\right| - \dfrac{1-\sqrt{5}}{2}\log\left|u-\dfrac{1-\sqrt{5}}{2}\right|\right\} = -\log|x| + C$

$\dfrac{1}{\sqrt{5}}\left\{\dfrac{1}{2}\log\left|\dfrac{2u-1-\sqrt{5}}{2u-1+\sqrt{5}}\right| + \dfrac{\sqrt{5}}{2}\log|u^2-u-1|\right\} = -\log|x| + C$

$\log\left|\dfrac{2u-1-\sqrt{5}}{2u-1+\sqrt{5}}\right| + \sqrt{5}\log|u^2-u-1| + 2\sqrt{5}\log|x| = C$

$\log\left|\dfrac{2u-1-\sqrt{5}}{2u-1+\sqrt{5}}\right|\left[|u^2-u-1||x^2|\right]^{\sqrt{5}} = C$

$\dfrac{2y-(1+\sqrt{5})x}{2y-(1-\sqrt{5})x}\,(y^2-yx-x^2)^{\sqrt{5}} = C \qquad \{2y-(1+\sqrt{5})x\}(y^2-xy-x^2)^{\sqrt{5}} = C\{2y-(1-\sqrt{5})x\}$

(2) $y' = (x+y)^2 \qquad u = x+y \qquad u' = 1+y'$

$u'-1 = u^2 \qquad u' = u^2+1 \qquad \dfrac{u'}{u^2+1} = 1 \qquad \tan^{-1}u = x + C$

$u = \tan(x+C) \qquad \therefore\ y = -x + \tan^{-1}(x+C)$

(3) $dx - y\,dy = x^2 y\,dy \qquad$ ($dx \to dy$ ミスプリ)

$dx = y(1+x^2)\,dy \qquad \dfrac{1}{1+x^2}\,dx = y\,dy \qquad \dfrac{1}{2}y^2 = \tan^{-1}x + C$

$\therefore\ y^2 = 2\tan^{-1}x + C$

1.5 ベルヌーイ

(1) $y' + \dfrac{1}{x}y = x^2y^2 \qquad \dfrac{y'}{y^2} + \dfrac{1}{x}\dfrac{1}{y} = x^2 \qquad z = \dfrac{1}{y},\ z' = \dfrac{-y'}{y^2}$

$-z' + \dfrac{1}{x}z = x^2 \qquad z' - \dfrac{1}{x}z = -x^2 \qquad \int\dfrac{1}{x}dx = \log|x|$

$z = x\left\{\int -x^2\dfrac{1}{x}\,dx + C\right\} = x\left(-\dfrac{x^2}{2} + C\right) = -\dfrac{x^3}{2} + Cx$

$y = \dfrac{2}{-x^3 + Cx}$

(2) $y' + \dfrac{1}{x}y = -\dfrac{x^2}{2}y^3 \qquad \dfrac{y'}{y^3} + \dfrac{1}{x}\dfrac{1}{y^2} = -\dfrac{x^2}{2} \qquad z = \dfrac{1}{y^2},\ z' = \dfrac{-2y'}{y^3}$

$-\dfrac{1}{2}z' + \dfrac{1}{x}z = -\dfrac{x^2}{2} \qquad z' - \dfrac{2}{x}z = x^2 \qquad \int\dfrac{2}{x}dx = 2\log x$

$z = x^2\left\{\int\dfrac{1}{x^2}x^2\,dx + C\right\} = x^3 + Cx^2$

$y^2 = \dfrac{1}{x^3 + Cx^2}$

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1.5 (3) $y' - \frac{1}{x+1}y = -y^2$ $\quad \frac{y'}{y^2} - \frac{1}{x+1}\frac{1}{y} = -1$ $\quad z = \frac{1}{y},\ z' = \frac{-y'}{y^2}$

$-z' - \frac{1}{x+1}z = -1$ $\quad z' + \frac{1}{x+1}z = 1$ $\quad \int \frac{1}{x+1}dx = \log|x+1|$

$z = \frac{1}{x+1}\{\int (x+1)dx + c\} = \frac{1}{x+1}\{\frac{1}{2}(x+1)^2 + c\}$

$\frac{1}{y} = \frac{(x+1)^2+c}{2(x+1)}$ $\quad \therefore\ y = \frac{2(x+1)}{(x+1)^2+c}$

(4) $y' = y(1+xy)$ $\quad \frac{y'}{y^2} - \frac{1}{y} = x$ $\quad z = \frac{1}{y}$ $\quad z' = \frac{-y'}{y^2}$

$-z' - z = x$ $\quad z' + z = -x$

$z = e^{-x}\{\int -xe^x dx + c\} = e^{-x}(-xe^x + e^x + c) = -x + 1 + ce^{-x}$

$y = \frac{1}{-x+1+ce^{-x}} = \frac{e^x}{(1-x)e^x + c}$

(5) $y' - \frac{1}{x-1}y + y^2 = 0$ $\quad \frac{y'}{y^2} - \frac{1}{x-1}\frac{1}{y} + 1 = 0$ $\quad z = \frac{1}{y}$ $\quad z' = \frac{-y'}{y^2}$

$-z' - \frac{1}{x-1}z = -1$ $\quad z' + \frac{1}{x-1}z = 1$ $\quad \int \frac{1}{x-1}dx = \log|x-1|$

$z = \frac{1}{x-1}\{\int (x-1)dx + c\} = \frac{1}{x-1}\{\frac{1}{2}(x-1)^2 + c\} = \frac{(x-1)^2+a}{2(x-1)}$

$y = \frac{2(x-1)}{(x-1)^2+a}$

1.6 (1) $y' + y(y-1) = 0$ $\quad \frac{y'}{y(y-1)} = -1$ $\quad -(\frac{1}{y} - \frac{1}{y-1})y' = -1$

$\log \frac{y}{y-1} = x + c$ $\quad \frac{y}{y-1} = ae^x$ $\quad y = yae^x - ae^x$

$y = \frac{ae^x}{ae^x - 1} = \frac{e^x}{e^x - c}$

(2) $\int_0^1 y\,dx = y_0$ $\quad \therefore\ [\log|e^x - c|]_0^1 = y_0$ $\quad \log|\frac{e-c}{1-c}| = y_0$

$e - c = (1-c)e^{y_0}$ $\quad c(e^{y_0} - 1) = e^{y_0} - e$ $\quad c = \frac{e^{y_0} - e}{e^{y_0} - 1}$

$y = \frac{e^x}{e^x - \frac{e^{y_0} - e}{e^{y_0} - 1}} = \frac{e^x(e^{y_0} - 1)}{e^x(e^{y_0} - 1) - e^{y_0} + e}$

1.7 $y' + y = f(t)$

(1) $f(t) = 0$ のとき $y' + y = 0$ $\quad \frac{y'}{y} = -1$ $\quad \log|y| = -t + a$

$y = ce^{-t}$

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1.7 (2) $f(t)=\cos\omega t$ のとき $y=A\cos\omega t+B\sin\omega t$ とおいて

$-A\omega\sin\omega t+B\omega\cos\omega t+A\cos\omega t+B\sin\omega t=\cos\omega t$

$(A+B\omega)\cos\omega t+(B-A\omega)\sin\omega t=\cos\omega t$

$\therefore A+B\omega=1 \quad B-A\omega=0 \quad \therefore A+A\omega^2=1 \quad A=\frac{1}{1+\omega^2} \quad B=\frac{\omega}{1+\omega^2}$

$\therefore$ 特殊解 $y=\frac{1}{1+\omega^2}(\cos\omega t+\omega\sin\omega t)$

(3) 一般解 $y=\frac{1}{1+\omega^2}(\cos\omega t+\omega\sin\omega t)+Ce^{-t}$

(4) $f(t)=\cos t+\cos 3t$ のとき

特殊解 $y=\frac{1}{2}(\cos t+\sin t)+\frac{1}{10}(\cos 3t+3\sin 3t)$

$\cos 3t=\cos 2t\cos t-\sin 2t\sin t=(2\cos^2 t-1)\cos t-2\cos t\sin^2 t$

$=2\cos^3 t-\cos t-2\cos t(1-\cos^2 t)=4\cos^3 t-3\cos t$

$\sin 3t=\sin 2t\cos t+\cos 2t\sin t=2\sin t\cos t+(1-2\sin^2 t)\sin t$

$=2\sin t(1-\sin^2 t)+\sin t-2\sin^3 t=3\sin t-4\sin^3 t$

$\frac{1}{5}\cos t+\frac{7}{5}\sin t+\frac{2}{5}\cos^3 t-\frac{6}{5}\sin^3 t$

1.8 $y'=1/\log(2x+y+3)-2$ $\quad 2x+y+3=z$ とおくと $z'=2+y'$

$y'+2=1/\log(2x+y+3) \quad z'-2+2=1/\log z \quad z'=1/\log z$

$z'\log z=1 \quad \int\log z\,dz=\int dx \quad z\log z-z=x+C$

$(2x+y+3)\log(2x+y+3)=3x+y+3+C$

1.9 $(p^2+1)^2-(px-y)^2=0 \quad p=y'$

$(p^2+1-px+y)(p^2+1+px-y)=0$

$p^2-px+y+1=0 \quad p^2+px-y+1=0$ クレーロー

$2pp'-p'x-p+p=0 \quad 2pp'+p'x+p-p=0$ (xで微分して)

$p'(2p-x)=0 \quad p'(2p+x)=0$

$\therefore p'=0 \ \therefore p=C$ or $p=\frac{x}{2} \ \therefore -\frac{x^2}{4}+1+y=0$ or $p=-\frac{x}{2} \ -\frac{x^2}{4}-y+1=0$

$\underline{4y=x^2-4} \qquad \underline{4y=-x^2+4}$

$\therefore (C^2+1)^2-(Cx-y)^2=0$

1.10 (1) $\begin{vmatrix}1 & 2x^2 & 2x & 2\\ 2x^2 & 1 & x & x^2\\ 2x & x & y' & x\\ 2 & x^2 & x & 1\end{vmatrix}=0 \quad \begin{vmatrix}-3 & 0 & 2x & 2\\ 0 & 1-x^2 & x & x^2\\ 0 & x-xy' & y' & x\\ 0 & 0 & x & 1\end{vmatrix}=0 \quad \begin{vmatrix}1-x^2 & 0 & x^2-1\\ x-xy' & y' & x\\ 0 & x & 1\end{vmatrix}=0 \quad \begin{vmatrix}1-x^2 & 0 & 0\\ x(1-y') & y' & 2x-xy'\\ 0 & x & 1\end{vmatrix}=0$

$(1-x^2)\{y'-x(2x-xy')\}=0 \quad (1-x^2)\{(1+x^2)y'-2x^2\}=0$

$\therefore y'=\frac{2x^2}{1+x^2}=2-\frac{2}{1+x^2}$

(2) $y=2x-2\tan^{-1}x+C$

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1.11 $x'=-kx$ $\quad x(0)=2$ $\quad J(k)=\int_0^\infty (1+k^2)\,x\,dt$ を最小

$\frac{1}{x}dx=-k\,dt$ $\quad \log|x|=-kt+c$

$x=ae^{-kt}$ $\quad x(0)=a=2$ $\quad$ ∴ $x=2e^{-kt}$

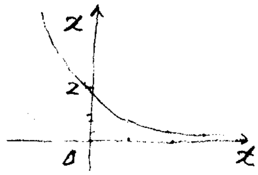
∴ $J(k)=\int_0^\infty(1+k^2)2e^{-kt}dt = \left[\frac{2(1+k^2)}{-k}e^{-kt}\right]_0^\infty$

∴ $J(k)=\begin{cases}\frac{2(1+k^2)}{k} & k>0\\ \infty & k<0\end{cases}$

$J'(k)=2\left(\frac{1}{k}+k\right)'=2\left(-\frac{1}{k^2}+1\right)=\frac{2(1-k)(k+1)}{k^2}$

$0<k<1$ 増加 $\quad k>1$ 減少

$k=1$ のとき最大値 $J(1)=2$ をとる $\quad x(t)=2e^{-t}$



1.12 $\quad \frac{dy}{dx}=(4x+y+2)^2$

(1) $u=4x+y+2$ とおくと $\quad u'=4+y'$ $\quad$ ∴ $y'=u'-4$

$u'-4=u^2$ $\quad u'=u^2+4$

(2) $\frac{u'}{u^2+4}=1$ $\quad \frac{1}{2}\tan^{-1}\frac{u}{2}=x+c$ $\quad \frac{u}{2}=\tan(2x+2c)$

∴ $4x+y+2=2\tan(2x+a)$ $\quad y=2\tan(2x+a)-4x-2$

(3) $x=0$ のとき $y=0$

$0=2\tan a-2=0$ $\quad \tan a=1$ $\quad a=\frac{\pi}{4}$

∴ $y=2\tan\left(2x+\frac{\pi}{4}\right)-4x-2$

1.13 $y=A\sin mx+B\cos mx$ $\quad y'=Am\cos mx-Bm\sin mx$

$y''=-Am^2\sin mx-Bm^2\cos mx$

∴ $y''+m^2y=0$

1.14 $\quad xy'=2y-x$

(1) $y=xu$ $\quad y'=u+xu'$ $\quad$ ∴ $x(u+xu')=2ux-x$

$u+xu'=2u-1$ $\quad xu'=u-1$

(2) $x=e^t$ $\quad t=\log x$ $\quad \frac{dt}{dx}=\frac{1}{x}$

$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{1}{x}\frac{dy}{dt}$

$x\frac{1}{x}\frac{dy}{dt}=2y-e^t$ $\quad \frac{dy}{dt}-2y=-e^t$ $\quad$ 線形

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1.14 (3) $y(3)=0$ (1)より $\frac{u'}{u-1}=\frac{1}{x}$ ∴ $\log|u-1|=\log|x|+a$

∴ $u-1=Cx$ $ux=Cx^2+x$ $y=Cx^2+x$

$y(3)=0$ より $9C+3=0$ $C=-\frac{1}{3}$

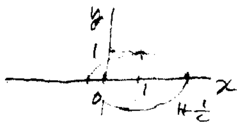
∴ $y=-\frac{1}{3}x^2+x$

1.15 $dx+xy\,dy=y^2dx+y\,dy$ $(y^2-1)dx+y(1-x)dy=0$

$\frac{y}{y^2-1}dy-\frac{1}{x-1}dx=0$ $\frac{1}{2}\log|y^2-1|-\log|x-1|=C$

∴ $y^2-1=a(x-1)^2$ $a<0$ のとき $a=-c^2$ とおくと

$c^2(x-1)^2+y^2=1$ だ円



1.16 $y=y(x)$ $y'=f(x,y)$, $y''=f_x(x,y)+f_y(x,y)y'$

∴ $y''=f_x(x,y)+f_y(x,y)\cdot f(x,y)$

$y'''=f_{xx}(x,y)+f_{xy}(x,y)f(x,y)+f_{yx}(x,y)\cdot f(x,y)+f_{yy}(x,y)\cdot\{f(x,y)\}^2$

$+f_y(x,y)f_x(x,y)+\{f_y(x,y)\}^2f(x,y)$

$=f_{xx}(x,y)+2f_{xy}(x,y)f(x,y)+f_{yy}(x,y)\{f(x,y)\}^2+f_x(x,y)f_y(x,y)$

$+\{f_y(x,y)\}^2f(x,y)$

1.17 $y'=y^2-1$ $y(0)=c$

$\frac{y'}{y^2-1}=1$, $\frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}\right)=1$ $\frac{1}{2}\log\left|\frac{y-1}{y+1}\right|=x+a$

$\frac{y-1}{y+1}=be^{2x}$ $x=0$ のとき $y=c$ $\frac{c-1}{c+1}=b$

$y-1=(y+1)be^{2x}$ $(1-be^{2x})y=be^{2x}+1$

$y=\frac{1+be^{2x}}{1-be^{2x}}=\frac{c+1+(c-1)e^{2x}}{c+1-(c-1)e^{2x}}$

$c=-1$ のとき $y=\frac{-2e^{2x}}{2e^{2x}}=-1$

$c=0$ のとき $y=\frac{1-e^{2x}}{1+e^{2x}}$

$c=1$ のとき $y=\frac{2}{2}=1$

$c=2$ のとき $y=\frac{3+e^{2x}}{3-e^{2x}}$

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1.18 $y' + xy = x^2 - x + 1$

$y = x^m$ を代入すると

$mx^{m-1} + x^{m+1} = x^2 - x + 1$ $\therefore$ $m+1 \leqq 2$ $\therefore$ $m \leqq 1$

$\therefore$ 特殊解は一次式 $y = ax + b$ である

$a + ax^2 + bx = x^2 - x + 1$ $\therefore$ $a = 1$, $b - 1$ $\therefore$ $y = x - 1$

1.19 $2x^2y' - x^2y^2 + 2xy + 1 = 0$

(1) $u = xy$ とおくと $u' = y + xy'$ $\therefore$ $xu' = xy + x^2y'$

$\therefore$ $x^2y' = xu' - u$

$\therefore$ $2(xu' - u) - u^2 + 2u + 1 = 0$

$2xu' - u^2 + 1 = 0$ $\underline{2xu' = u^2 - 1}$

(2) $\dfrac{2u'}{u^2-1} = \dfrac{1}{x}$ $\left(\dfrac{1}{u-1} - \dfrac{1}{u+1}\right)du = \dfrac{1}{x}dx$

$\log\left|\dfrac{u-1}{u+1}\right| = \log|x| + a$

$\dfrac{u-1}{u+1} = cx$ $u - 1 = cxu + cx$

$(1 - cx)u = cx + 1$ $u = \dfrac{1+cx}{1-cx}$

$\therefore$ $y = \dfrac{1+cx}{x(1-cx)}$

P.69

§2 定係数の線形微分方程式 (1)

2.1 (1) $y''' - 3y'' + 4y' - 12y = 0$

$$\begin{array}{rrrr|r} 1 & -3 & 4 & -12 & 3 \\ & 3 & 0 & 12 & \\ \hline 1 & 0 & 4 & 0 & \end{array}$$

特性方程式 $t^3 - 3t^2 + 4t - 12 = 0$

$(t-3)(t^2+4)=0 \quad \therefore t = 3,\ \pm 2i$

$y = C_1 e^{3x} + C_2 \cos 2x + C_3 \sin 2x$

(2) $a^2 y^{(4)} - y'' = 0$

特性方程式 $a^2 t^4 - t^2 = 0 \quad t^2(a^2t^2 - 1) = 0,\ t = 0,\ \frac{1}{a}, -\frac{1}{a}$

$y = C_1 + C_2 x + C_3 e^{\frac{1}{a}x} + C_4 e^{-\frac{1}{a}x}$

(3) $y''' + y'' + 4y = 0$

$$\begin{array}{rrrr|r} 1 & 1 & 0 & 4 & -2 \\ & -2 & 2 & -4 & \\ \hline 1 & -1 & 2 & 0 & \end{array}$$

特性方程式 $t^3 + t^2 + 4 = 0$

$(t+2)(t^2 - t + 2) = 0 \quad t = -2,\ \frac{1 \pm \sqrt{7}i}{2}$

$y = C_1 e^{-2x} + e^{\frac{1}{2}x}\left(C_2 \cos \frac{\sqrt{7}}{2}x + C_3 \sin \frac{\sqrt{7}}{2}x\right)$

(4) $y'' - y' - 2y = 0 \quad t^2 - t - 2 = 0 \quad (t-2)(t+1) = 0$
$t = 2,\ -1$

$y = C_1 e^{2x} + C_2 e^{-x}$

(5) $y'' - 5y' + 5y = 0 \quad t^2 - 5t + 5 = 0 \quad t = \frac{5 \pm \sqrt{5}}{2}$

$y = C_1 e^{\frac{5+\sqrt{5}}{2}x} + C_2 e^{\frac{5-\sqrt{5}}{2}x}$

(6) $y'' + 2y' - 3y = 0 \quad t^2 + 2t - 3 = 0 \quad t = 1,\ -3$

$y = C_1 e^{x} + C_2 e^{-3x}$

2.2 (1) $y''' - y = 0 \quad y(\infty) = 0$

$t^3 - 1 = 0 \quad t = 1,\ \frac{-1 \pm \sqrt{3}i}{2}$

$\therefore y = C_1 e^{x} + e^{-\frac{x}{2}}\left(C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x\right)$

$x \to \infty$ のとき $y \to 0 \quad \therefore C_1 = 0$

$\therefore y = e^{-\frac{x}{2}}\left(C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x\right)$

(2) $x'' + 3x' + 2x = 0 \quad x(0) = 1,\ x'(0) = 2$

$t^2 + 3t + 2 = 0 \quad t = -1,\ -2$

$x = C_1 e^{-t} + C_2 e^{-2t} \quad C_1 + C_2 = 1 \quad -C_1 - 2C_2 = 2 \quad C_2 = -3 \quad C_1 = 4$

$\therefore x = 4e^{-t} - 3e^{-2t}$

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2.2 (3) $y''-2ay'+a^2=0$; $y(0)=0$ $y'(0)=1$

特性方程式 $t^2-2at+a^2=0$ $t=a$

$y=(c_1+c_2x)e^{ax}$ $y'=(c_2+c_1a+c_2ax)e^{ax}$

$c_1=0$ $1=c_2+c_1a$ $c_2=1$

$\therefore y=xe^{ax}$

(4) $y''+3y'+2y=0$, $y(0)=1$ $y'(0)=1$

$t^2+3t+2=0$ $t=-1,-2$

$y=c_1e^{-x}+c_2e^{-2x}$ $y'=-c_1e^{-x}-2c_2e^{-2x}$

$1=c_1+c_2$ $1=-c_1-2c_2$ $c_2=-2$ $c_1=3$

$y=3e^{-x}-2e^{-2x}$

2.3 (1) $y''+k^2y=0$ $t^2+k^2=0$ $t=\pm ki$

$y=c_1\cos kx+c_2\sin kx$, $y'=-kc_1\sin kx+kc_2\cos kx$

$y(0)=0$, $y'(0)=k$ $y(1)=\frac{1}{\sqrt{2}}$

$c_1=0$ $kc_2=k$ $c_2=1$ $\frac{1}{\sqrt{2}}=\sin k$ $k=n\pi+(-1)^n\frac{\pi}{4}$

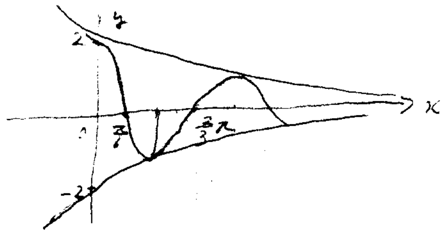
2.4 $y''+2y'+5y=0$ $y(0)=\sqrt{3}$ $y'(0)=-\sqrt{3}-2$

$t^2+2t+5=0$ $t=-1\pm 2i$

$y=e^{-x}(c_1\cos 2x+c_2\sin 2x)$ $y'=e^{-x}\{(-c_1+2c_2)\cos 2x+(-2c_1-c_2)\sin 2x\}$

$c_1=\sqrt{3}$ $-c_1+2c_2=-\sqrt{3}-2$ $c_2=-1$

$y=e^{-x}(\sqrt{3}\cos 2x-\sin 2x)=2e^{-x}\cos(2x+\frac{\pi}{6})$



2.5 $y''+y=0$ $y(0)=0$ $y'(0)=1$

$t^2+1=0$ $t=\pm i$

$y=c_1\cos x+c_2\sin x$ $y'=-c_1\sin x+c_2\cos x$

$c_1=0$ $c_2=1$

$y=\sin x$ $y(\frac{\pi}{2})=1$

P.70

§3 定係数の線形微分方程式(2)

3.1 演算子法

(1) $(D^2-3D+2)y=x$ $\quad (D-1)(D-2)y=x$ $\quad y=\dfrac{1}{2(1-D)(1-\frac{D}{2})}x=\dfrac{1}{2}(1+D)(1+\dfrac{D}{2})x$

$\therefore y=C_1e^{x}+C_2e^{2x}+\dfrac{1}{2}x+\dfrac{3}{4}$ $\quad =\dfrac{1}{2}(1+\dfrac{3}{2}D)x=\dfrac{x}{2}+\dfrac{3}{4}$

3.2

(1) $y''+y'-y=xe^{2x}$ $\quad (D^2+D-1)y=xe^{2x}$, $\quad y=\dfrac{1}{D^2+D-1}xe^{2x}=e^{2x}\dfrac{1}{(D+2)^2+D+2-1}x$

$\therefore y=e^{2x}\dfrac{1}{D^2+5D+5}x=e^{2x}\dfrac{1}{5}(1-D)x=e^{2x}\dfrac{1}{5}(x-1)$

補助方程式 $t^2+t-1=0$ $\quad t=\dfrac{-1\pm\sqrt{5}}{2}$

$\therefore y=C_1e^{\frac{-1+\sqrt{5}}{2}x}+C_2e^{\frac{-1-\sqrt{5}}{2}x}+\dfrac{1}{5}e^{2x}(x-1)$

(2) $(D^2-2D+2)y=x^2+1$ $\quad y=\dfrac{1}{2(1-(D-\frac{D^2}{2}))}(x^2+1)=\dfrac{1}{2}\{1+D-\dfrac{D^2}{2}+D^2\}(x^2+1)$

補助方程式 $\quad =\dfrac{1}{2}(x^2+1+2x+1)=\dfrac{1}{2}(x^2+2x+2)$

$t^2-2t+2=0$ $\quad t=1\pm i$

$\therefore y=e^{x}(C_1\cos x+C_2\sin x)+\dfrac{1}{2}(x^2+2x+2)$

(3) $y''+y=e^{x}$ $\quad (D^2+1)y=e^{x}$ $\quad y=\dfrac{1}{D^2+1}e^{x}=\dfrac{1}{2}e^{x}$

$t^2+1=0$ $\quad t=\pm i$

$y=C_1\cos x+C_2\sin x+\dfrac{1}{2}e^{x}$

(4) $y''+2y'+3y=x$ $\quad (D^2+2D+3)y=x$ $\quad y=\dfrac{1}{3(1+\frac{2}{3}D+\frac{1}{3}D^2)}x=\dfrac{1}{3}(1-\dfrac{2}{3}D)x$

$t^2+2t+3=0$ $\quad =\dfrac{1}{3}(x-\dfrac{2}{3})$

$t=-1\pm\sqrt{2}i$

$y=e^{-x}(C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x)+\dfrac{1}{3}(x-\dfrac{2}{3})$

(5) $2y''+4y'+y=e^{-2x}$, $\quad (2D^2+4D+1)y=e^{-2x}$, $\quad y=\dfrac{1}{2D^2+4D+1}e^{-2x}=\dfrac{e^{-2x}}{1}$

$2t^2+4t+1=0$ $\quad t=\dfrac{-2\pm\sqrt{2}}{2}=-1\pm\dfrac{1}{\sqrt{2}}$

$\therefore y=e^{-x}(C_1e^{\frac{1}{\sqrt{2}}x}+C_2e^{-\frac{1}{\sqrt{2}}x})+e^{-2x}$

(6) $y''+y'+y=e^{x}$ $\quad y=\dfrac{1}{D^2+D+1}e^{x}=\dfrac{1}{3}e^{x}$

$t^2+t+1=0$ $\quad t=-\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$

$y=\dfrac{1}{3}e^{x}+e^{-\frac{1}{2}x}(C_1\cos\dfrac{\sqrt{3}}{2}x+C_2\sin\dfrac{\sqrt{3}}{2}x)$

P.70

3.2 (7) $y'' - 4y' + 8y = e^{2x}$ $\quad y = \frac{1}{D^2-4D+8} e^{2x} = \frac{1}{4} e^{2x}$

$t^2 - 4t + 8 = 0 \quad t = 2 \pm 2i$

$y = \frac{1}{4} e^{2x} + e^{2x}(C_1 \cos 2x + C_2 \sin 2x)$

(8) $y'' + 4y' + 13y = 9e^{4x}$ $\quad y = \frac{9}{D^2+4D+13} e^{4x} = \frac{9}{45} e^{4x} = \frac{1}{5} e^{4x}$

$t^2 + 4t + 13 = 0 \quad t = -2 \pm 3i$

$y = \frac{1}{5} e^{4x} + e^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$

(9) $y'' - 6y' + 8y = e^x$ $\quad y = \frac{1}{D^2-6D+8} e^x = \frac{e^x}{3}$

$t^2 - 6t + 8 = 0, \quad (t-2)(t-4) = 0 \quad t = 2, 4$

$\therefore y = C_1 e^{2x} + C_2 e^{4x} + \frac{1}{3} e^x$

(10) $y'' - y' - 2y = x + 1$ $\quad y = \frac{1}{-2(1 + \frac{D}{2} - \frac{D^2}{2})}(x+1) = \frac{-1}{2}\left(1 - \frac{D}{2}\right)(x+1)$

$t^2 - t - 2 = 0$ $\quad = -\frac{1}{2}\left(x + 1 - \frac{1}{2}\right) = -\frac{1}{2}\left(x + \frac{1}{2}\right)$

$(t+1)(t-2) = 0$

$\therefore y = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{2}x - \frac{1}{4}$

(11) $y'' - y' - 2y = x^2 + x$ $\quad y = \frac{1}{-2(1 + \frac{D}{2} - \frac{D^2}{2})}(x^2+x) = -\frac{1}{2}\left(1 - \frac{D}{2} + \frac{D^2}{2} + \frac{D^2}{4}\right)(x^2+x)$

$t^2 - t - 2 = 0$ $\quad = -\frac{1}{2}\left(x^2 + x - \frac{1}{2}(2x+1) + \frac{3}{4}\cdot 2\right)$

$t = -1, 2$ $\quad = -\frac{1}{2}(x^2+1)$

$\therefore y = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{2}(x^2+1)$

(12) $y'' + y = e^x + 1$ $\quad y = \frac{1}{D^2+1}(e^x+1) = \frac{1}{2} e^x + 1$

$t^2 + 1 = 0 \quad t = \pm i$

$\therefore y = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^x + 1$

(13) $y'' + 4y' + 13y = x$ $\quad y = \frac{1}{13 + 4D + D^2} x = \frac{1}{13}\left(1 - \frac{4}{13} D\right) x$

$t^2 + 4t + 13 = 0$ $\quad = \frac{1}{13}\left(x - \frac{4}{13}\right)$

$t = -2 \pm 3i$

$\therefore y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x) + \frac{1}{13}\left(x - \frac{4}{13}\right)$

P.76

3.3 (1) $y''+y'-6y=\sin x$ $(D^2+D-6)y=\sin x$

$\lambda^2+\lambda-6=0$ $(\lambda+3)(\lambda-2)=0$ $\lambda=2,-3$

$y=A\cos x+B\sin x$ とおく $y'=B\cos x-A\sin x$, $y''=-A\cos x-B\sin x$

$(-A+B-6A)\cos x+(-B-A-6B)\sin x=\sin x$

$-7A+B=0$ $-A-7B=1$ $A=-\frac{1}{50}$ $B=-\frac{7}{50}$

$\therefore y=C_1e^{2x}+C_2e^{-3x}-\frac{1}{50}(\cos x+7\sin x)$

(2) $y''+y'+2y=\sin x$ $(D^2+D+2)y=\sin x$

$\lambda^2+\lambda+2=0$ $\lambda=-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$

$y=A\cos x+B\sin x$ とおく $y'=B\cos x-A\sin x$ $y''=-A\cos x-B\sin x$

$(-A+B+2A)\cos x+(-B-A+2B)\sin x=\sin x$

$A+B=0$ $B-A=1$ $B=\frac{1}{2}$ $A=-\frac{1}{2}$

$\therefore y=\frac{1}{2}(-\cos x+\sin x)+e^{-\frac{1}{2}x}(C_1\cos\frac{\sqrt{7}}{2}x+C_2\sin\frac{\sqrt{7}}{2}x)$

(3) $y''-2y'+5y=\sin x$ $(D^2-2D+5)y=\sin x$

$\lambda^2-2\lambda+5=0$ $\lambda=1\pm2i$

$y=A\cos x+B\sin x$ とおくと $y'=B\cos x-A\sin x$, $y''=-A\cos x-B\sin x$

$(-A-2B+5A)\cos x+(-B+2A+5B)\sin x=\sin x$

$4A-2B=0$ $2A+4B=1$ $A=\frac{1}{10}$, $B=\frac{1}{5}$

$\therefore y=e^{x}(C_1\cos 2x+C_2\sin 2x)+\frac{1}{10}(\cos x+2\sin x)$

(4) $y''-2y'+y=4\sin x$ $(D^2-2D+1)y=4\sin x$

$\lambda^2-2\lambda+1=0$ $\lambda=1$,

$y=A\cos x+B\sin x$ とおくと $y'=B\cos x-A\sin x$, $y''=-A\cos x-B\sin x$

$(-A-2B+A)\cos x+(-B+2A+B)\sin x=4\sin x$

$-2B=0$ $2A=4$ $B=0$, $A=2$

$\therefore y=2\cos x+(C_1+C_2x)e^{x}$

(5) $y''-5y'+4y=\frac{1}{n^2}\cos nx$ $(D^2-5D+4)y=\frac{1}{n^2}\cos nx$

$\lambda^2-5\lambda+4=0$, $(\lambda-1)(\lambda-4)=0$ $\lambda=1,4$

$y=A\cos nx+B\sin nx$ とおくと $y'=nB\cos nx-nA\sin nx$, $y''=-n^2A\cos nx-n^2B\sin nx$

P.70

3.3 (5) $(-n^2A - 5nB + 4A)\cos nx + (-n^2B + 5nA + 4B)\sin nx = \frac{1}{n^2}\cos nx$

$(4-n^2)A - 5nB = \frac{1}{n^2}$ $5nA + (4-n^2)B = 0$ $A = -\frac{4-n^2}{5n}B$

$-\frac{(4-n^2)^2+25n^2}{5n}B = \frac{1}{n^2}$ $B = -\frac{5n}{\{(4-n^2)^2+25n^2\}n^2} = -\frac{5}{n(n^4+17n^2+16)}$

$A = \frac{4-n^2}{n^2(n^4+17n^2+16)}$

$\therefore y = C_1e^{x} + C_2e^{4x} + \frac{1}{n^2(n^2+1)(n^2+16)}\{(n^2-4)\cos nx - 5n\sin nx\}$

(6) $y''+2y'+5y=\sin x$ $(D^2+2D+5)y=\sin x$

$t^2+2t+5=0$ $t=-1\pm 2i$

$y=A\cos x+B\sin x$ とおくと $y'=B\cos x - A\sin x$, $y''=-A\cos x - B\sin x$

$(-A+2B+5A)\cos x + (-B-2A+5B)\sin x = \sin x$

$4A+2B=0$ $4B-2A=1$ $B=\frac{2}{10}$ $A=-\frac{1}{10}$

$\therefore y=e^{-x}(C_1\cos 2x + C_2\sin 2x)+\frac{1}{10}(-\cos x+2\sin x)$

(7) $y''+3y'+2y=\cos x$ $(D^2+3D+2)y=\cos x$

$t^2+3t+2=0$ $t=-1, -2$

$y=A\cos x+B\sin x$ とおくと $y'=B\cos x - A\sin x$, $y''=-A\cos x-B\sin x$

$(-A+3B+2A)\cos x+(-B-3A+2B)\sin x=\cos x$

$A+3B=1$ $-3A+B=0$ $A=\frac{1}{10}$, $B=\frac{3}{10}$

$\therefore y=C_1e^{-x}+C_2e^{-2x}+\frac{1}{10}(\cos x+3\sin x)$

3.4 (1) $y''-2y'+2y=e^{-x}\cos x$, $(D^2-2D+2)y=e^{-x}\cos x$

$t^2-2t+2=0$ $t=1\pm i$

与式は $(D^2-2D+2)y=e^{(-1+i)x}$ の実部

$y=\frac{1}{D^2-2D+2}e^{(-1+i)x}=\frac{1}{(-1+i)^2-2(-1+i)+2}e^{(-1+i)x}=\frac{1}{4(1-i)}e^{(-1+i)x}$

$=\frac{1+i}{8}(\cos x+i\sin x)e^{-x}=\frac{e^{-x}}{8}(\cos x-\sin x)+\frac{e^{-x}}{8}(\cos x+\sin x)i$

$\therefore$ 特殊解 $\frac{1}{8}e^{-x}(\cos x-\sin x)$

$\therefore y=e^{x}(C_1\cos x+C_2\sin x)+\frac{e^{-x}}{8}(\cos x-\sin x)$

P.71

3.5 (1) $y''-3y'+2y=e^x \quad (D^2-3D+2)y=e^x$

$t^2-3t+2=0 \quad t=1,2 \quad y=\frac{1}{(D-1)(D-2)}e^x=\frac{1}{(D-1)(-1)}e^x$

$=-e^x\frac{1}{D}\cdot 1=-xe^x$

$\therefore y=C_1e^x+C_2e^{2x}-xe^x$

(2) $y''-2y'+5y=e^x\cos 2x \quad (D^2-2D+5)y=e^x\cos 2x$

$t^2-2t+5=0 \quad t=1\pm 2i \quad (D^2-2D+5)y=e^{(1+2i)x}$ の実部

$y=\frac{1}{(D-1-2i)(D-1+2i)}e^{(1+2i)x}=\frac{1}{D-1-2i}\frac{1}{4i}e^{(1+2i)x}$

$=\frac{1}{4i}e^{(1+2i)x}\frac{1}{D}1=\frac{1}{4i}e^{(1+2i)x}x$

$=\frac{-i}{4}(\cos 2x+i\sin 2x)e^x x=\frac{xe^x}{4}(\sin 2x-i\cos 2x)$

$\therefore$ 特殊解 $\frac{xe^x}{4}\sin 2x$

$y=e^x(C_1\cos 2x+C_2\sin 2x)+\frac{1}{4}xe^x\sin 2x$

(3) $y''+y'-2y=e^x \quad (D^2+D-2)y=e^x$

$t^2+t-2=0 \quad t=1,-2$

$y=\frac{1}{(D-1)(D+2)}e^x=\frac{1}{3}\frac{1}{D-1}e^x=\frac{e^x}{3}\frac{1}{D}1=\frac{x}{3}e^x$

$y=C_1e^x+C_2e^{-2x}+\frac{x}{3}e^x$

(4) $y''-5y'+6y=e^x \quad (D^2-5D+6)y=e^x$

$t^2-5t+6=0 \quad t=2,3 \quad y=\frac{1}{D^2-5D+6}e^x=\frac{1}{2}e^x$

$y=C_1e^{2x}+C_2e^{3x}+\frac{1}{2}e^x$

(5) $y''+2y'-3y=e^x \quad (D^2+2D-3)y=e^x$

$t^2+2t-3=0,\ t=1\ -3. \quad y=\frac{1}{(D-1)(D+3)}e^x=\frac{e^x}{4}\frac{1}{D}=\frac{x}{4}e^x$

$y=C_1e^x+C_2e^{-3x}+\frac{x}{4}e^x$

(6) $y''+2y'-8y=e^{2x} \quad (D^2+2D-8)y=e^{2x}$

$t^2+2t-8=0 \quad t=2,-4. \quad y=\frac{1}{(D-2)(D+4)}e^{2x}=\frac{e^{2x}}{6}\frac{1}{D}\cdot 1=\frac{x}{6}e^{2x}$

$y=C_1e^{2x}+C_2e^{-4x}+\frac{1}{6}xe^{2x}$

(7) $y''-2y'-8y=e^{-2x} \quad (D^2-2D-8)y=e^{-2x}$

$t^2-2t-8=0 \quad t=4,-2 \quad y=\frac{1}{(D+2)(D-4)}e^{-2x}=\frac{e^{-2x}}{-6}\frac{1}{D}=-\frac{x}{6}e^{-2x}$

$y=C_1e^{4x}+C_2e^{-2x}-\frac{x}{6}e^{-2x}$

P. 71

3.6 (1) $y''-3y'+2y=x+e^{2x}\cos x$, $(D^2-3D+2)y=x+e^{2x}\cos x$

$t^2-3t+2=0 \quad t=1,2 \qquad y=\frac{1}{(D-1)(D-2)}x=\frac{1}{2}(1+D)(1+\frac{D}{2})x$

$=\frac{1}{2}(x+\frac{3}{2})$

$\frac{1}{(D-1)(D-2)}e^{(2+i)x}=\frac{1}{(1+i)i}e^{(2+i)x}=\frac{1+i}{-2}(\cos x+i\sin x)e^{2x}$ の実部

$=-\frac{1}{2}e^{2x}(\cos x-\sin x)-\frac{e^{2x}}{2}(\cos x+\sin x)i$

$\therefore y=C_1e^x+C_2e^{2x}+\frac{1}{2}x+\frac{3}{4}+\frac{e^{2x}}{2}(-\cos x+\sin x)$

(2) $y''-3y'+2y=x^2+e^x\sin x \quad (D^2-3D+2)y=x^2+e^x\sin x$

$t^2-3t+2=0 \quad t=1,2 \qquad y=\frac{1}{(D-1)(D-2)}x^2=\frac{1}{2}(1+D+D^2)(1+\frac{D}{2}+\frac{D^2}{4})x^2$

$=\frac{1}{2}(1+\frac{3}{2}D+\frac{7}{4}D^2)x^2=\frac{1}{2}(x^2+3x+\frac{7}{2})$

$\frac{1}{(D-1)(D-2)}e^{(1+i)x}=\frac{1}{i(-1+i)}e^{(1+i)x}=\frac{1-i}{-2}(\cos x+i\sin x)e^x$ の虚部

$=-\frac{e^x}{2}(\cos x+\sin x)-\frac{e^x}{2}(-\cos x+\sin x)i$

$y=C_1e^x+C_2e^{2x}+\frac{1}{2}(x^2+3x+\frac{7}{2})+\frac{e^x}{2}(\cos x-\sin x)$

(3) $y''-2y'+3y=3\sin x-\cos x$, $(D^2-2D+3)y=3\sin x-\cos x$

$t^2-2t+3=0 \quad t=1\pm\sqrt{2}i$

$y=A\cos x+B\sin x$ とおくと $y'=B\cos x-A\sin x \quad y''=-A\cos x-B\sin x$

$(-A-2B+3A)\cos x+(-B+2A+3B)\sin x=3\sin x-\cos x$

$2A-2B=-1 \quad 2A+2B=3 \quad A=\frac{1}{2} \quad B=1$

$y=e^x(C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x)+\frac{1}{2}\cos x+\sin x$

(4) $x''-x'-2x=e^{2t}+\sin t$, $(D^2-D-2)x=e^{2t}+\sin t$

$t^2-t-2=0 \quad t=-1,2 \qquad \frac{1}{(D-2)(D+1)}e^{2t}=\frac{1}{3}e^{2t}\frac{1}{D}1=\frac{t}{3}e^{2t}$

$\frac{1}{D^2-D-2}e^{it}=\frac{1}{-(3+i)}e^{it}=\frac{-3+i}{10}(\cos t+i\sin t)$ 虚部

$=\frac{1}{10}(-3\cos t-\sin t)+\frac{1}{10}(\cos t-3\sin t)i$

$\therefore y=C_1e^{-t}+C_2e^{2t}+\frac{t}{3}e^{2t}+\frac{1}{10}(\cos t-3\sin t)$

3.7 (1) $y'''-2y''-y'+2y=3e^{2x}$, $(D^3-2D^2-D+2)y=3e^{2x}$

$t^3-2t^2-t+2=0 \quad (t-2)(t^2-1)=0 \quad t=1,2,-1 \quad \frac{1}{(D-2)(D^2-1)}3e^{2x}=e^{2x}\frac{1}{D}1=xe^{2x}$

$y=C_1e^x+C_2e^{-x}+C_3e^{2x}+xe^{2x}$

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3.7 (2) $y'''+2y''-3y'=x^2e^{-x}+e^x\sin 2x$, $(D^3+2D^2-3D)y=x^2e^{-x}+e^x\sin 2x$

$t(t^2+2t-3)=0$ $t(t+3)(t-1)=0$, $t=0,1,-3$

$$\frac{1}{D(D+3)(D-1)}x^2e^{-x}=e^{-x}\frac{1}{(D-1)(D+2)(D-2)}x^2=e^{-x}(1+D+D^2)\frac{1}{4}(1+\frac{D^2}{4})x^2$$

$$=\frac{1}{4}e^{-x}(1+D+\frac{5}{4}D^2)x^2=\frac{1}{4}e^{-x}(x^2+2x+\frac{5}{2})$$

$$\frac{1}{D(D+3)(D-1)}e^{(1+2i)x}=\frac{1}{(1+2i)(4+2i)2i}e^{(1+2i)x}=\frac{-1}{20}e^x(\cos 2x+i\sin 2x)$$

虚部

$\therefore$ $y=C_1+C_2e^x+C_3e^{-3x}+\frac{e^{-x}}{4}(x^2+2x+\frac{5}{2})-\frac{e^x}{20}\sin 2x$

(3) $y'''-4y''+4y'=e^{2x}$, $(D^3-4D^2+4D)y=e^{2x}$

$t^3-4t^2+4t=0$ $t=0,2$ $\quad \frac{1}{D(D-2)^2}e^{2x}=\frac{e^{2x}}{2}\frac{1}{D^2}1=\frac{e^{2x}}{4}x^2$

$y=C_1+e^{2x}(C_2+C_3x)+\frac{x^2}{4}e^{2x}$

(4) $y'''-2y''+y'-2y=3e^{2x}$, $(D^3-2D^2+D-2)y=3e^{2x}$

$t^3-2t^2+t-2=0$ $(t-2)(t^2+1)=0$. $t=2,\ i,\ -i$

$$\frac{1}{(D-2)(D^2+1)}3e^{2x}=\frac{3}{5}e^{2x}\frac{1}{D}=\frac{3x}{5}e^{2x}$$

$y=C_1e^{2x}+C_2\cos x+C_3\sin x+\frac{3x}{5}e^{2x}$

3.8 (1) $y'''-y=0$; $y(\infty)=0$

$(D^3-1)y=0$ $\quad t^3-1=(t-1)(t^2+t+1)$ $\quad t=1,\ -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$y=C_3e^x+e^{-\frac{x}{2}}(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x)$

$x\to\infty$ のとき $y\to 0$ より $C_3=0$

$y=(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x)e^{-\frac{x}{2}}$

(2) $x''+3x'+2x=1$ $\quad x(0)=0$. $x'(0)=0$

$(D^2+3D+2)x=1$ $\quad t^2+3t+2=0$ $\quad t=-1,-2$

$\frac{1}{2+3D+D^2}1=\frac{1}{2}$

$\therefore$ $x=C_1e^{-t}+C_2e^{-2t}+\frac{1}{2}$ $\quad x'=-C_1e^{-t}-2C_2e^{-2t}$

$x(0)=C_1+C_2+\frac{1}{2}=0$ $\quad x'(0)=-C_1-2C_2=0$ $\quad C_1=-2C_2$

$C_2=\frac{1}{2}$ $\quad C_1=-1$

$x=-e^{-t}+\frac{1}{2}e^{-2t}+\frac{1}{2}$

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3.8 (3) $y''+3y'+2y=2$; $y(0)=0,\ y'(0)=0$

$(D^2+3D+2)y=2$ $\quad t^2+3t+2=0 \quad t=-1,\ -2$

$\dfrac{1}{D^2+3D+2}2=1$ $\quad \therefore\ y=C_1e^{-x}+C_2e^{-2x}+1 \quad y'=-C_1e^{-x}-2C_2e^{-2x}$

$0=C_1+C_2+1\ ,\ 0=-C_1-2C_2 \quad C_2=1 \quad C_1=-2$

$\therefore\ y=-2e^{-x}+e^{-2x}+1$

(4) $u''-u=e^{2t}$; $u(0)=a,\ u'(0)=b$

$(D^2-1)u=e^{2t} \quad x^2-1=0 \quad x=1,-1 \quad \dfrac{1}{D^2-1}e^{2t}=\dfrac{1}{3}e^{2t}$

$\therefore\ u=C_1e^{t}+C_2e^{-t}+\frac{1}{3}e^{2t} \quad u'=C_1e^{t}-C_2e^{-t}+\frac{2}{3}e^{2t}$

$u(0)=C_1+C_2+\frac{1}{3}=a \quad u'(0)=C_1-C_2+\frac{2}{3}=b$

$C_1=\frac{1}{2}(a+b-1) \quad C_2=\frac{1}{2}(a+b-1)+\frac{2}{3}-b=\frac{1}{2}(a-b+\frac{1}{3})$

$\therefore\ u=\frac{1}{2}(a+b-1)e^{t}+\frac{1}{6}(3a-3b+1)e^{-t}+\frac{1}{3}e^{2t}$

(5) $y''+k^2y=0$; $y(0)=1,\ y'(0)=2k$

$t^2+k^2=0 \quad t=\pm ki$

$y=C_1\cos kx+C_2\sin kx\ ,\ y'=-C_1k\sin kx+C_2k\cos kx$

$C_1=1 \quad C_2k=2k \quad C_2=2$

$\therefore\ y=\cos kx+2\sin kx$

3.9 $(D-\alpha)(D^2+2\xi D+1)y=e^{\beta x} \qquad 0<\xi<1$

補助方程式

$(t-\alpha)(t^2+2\xi t+1)=0 \qquad t=\alpha,\ t=-\xi\pm\sqrt{\xi^2-1}=-\xi\pm\sqrt{1-\xi^2}\,i$

$$\frac{1}{(D-\alpha)(D^2+2\xi D+1)}e^{\beta x}=\begin{cases}\dfrac{1}{(\beta-\alpha)(\beta^2+2\xi\beta+1)}e^{\beta x} & \alpha\neq\beta\\[2ex] \dfrac{x}{\beta^2+2\xi\beta+1}e^{\beta x} & \alpha=\beta\end{cases}$$

$\therefore$ $\alpha\neq\beta$ のとき $y=C_1e^{\alpha x}+e^{-\xi x}(C_2\cos\sqrt{1-\xi^2}\,x+C_3\sin\sqrt{1-\xi^2}\,x)+\dfrac{e^{\beta x}}{(\beta-\alpha)(\beta^2+2\xi\beta+1)}$

$\alpha=\beta$ のとき $y=C_1e^{\alpha x}+e^{-\xi x}(C_2\cos\sqrt{1-\xi^2}\,x+C_3\sin\sqrt{1-\xi^2}\,x)+\dfrac{xe^{\beta x}}{\beta^2+2\xi\beta+1}$

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3.10 $y''+(\alpha-2)y'+\alpha y=0$ 特性方程式 $t^2+(\alpha-2)t+\alpha=0$

(1) $t=\frac{2-\alpha}{2}\pm\frac{\sqrt{\alpha^2-8\alpha+4}}{2}$ これを a, b とするとき

$e^{ax}\to 0\ (x\to\infty)$ $e^{bx}\to 0\ (x\to\infty)$ となるためには

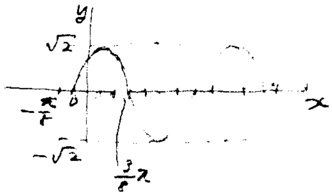
$a<0,\ b<0$ ∴ $-(\alpha-2)<0,\ \alpha>0$ ∴ $\alpha>2$

(2) $y''+2y=0$ $t^2+2=0$ $t=\pm\sqrt{2}i$

$y=C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x$ $y'=-\sqrt{2}C_1\sin\sqrt{2}x+\sqrt{2}C_2\cos\sqrt{2}x$

$y(0)=C_1=1$ $y'(0)=\sqrt{2}C_2=\sqrt{2}$ ∴ $C_1=C_2=1$

∴ $y=\cos\sqrt{2}x+\sin\sqrt{2}x=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$



3.11 $y''+y=\sin ax$ $(D^2+1)y=\sin ax$

(1) $t^2+1=0$ $t=\pm i$ $a\neq 1$ のとき $\frac{1}{D^2+1}\sin ax=\frac{1}{1-a^2}\sin ax$

$a=1$ のとき $\frac{1}{D^2+1}e^{aix}=\frac{1}{(D-i)(D+i)}e^{ix}$ 虚部

$=\frac{1}{2i}e^{ix}\frac{1}{D}=\frac{-ix}{2}(\cos x+i\sin x)$

$\frac{1}{D^2+1}\sin x=-\frac{x}{2}\cos x$

$a\neq 1$ のとき

$y=C_1\cos x+C_2\sin x+\frac{1}{1-a^2}\sin ax$

$a=1$ のとき

$y=C_1\cos x+C_2\sin x-\frac{x}{2}\cos x$

(2) $y=-\frac{x}{2}\cos x$

3.12 $y''-y=-2\sin x$ $y(0)=0$ $y'(0)=2$

$y=e^xu+\sin x$ とおくと $y'=e^xu+e^xu'+\cos x$

$y''=e^xu+2e^xu'+e^xu''-\sin x$

∴ $2e^xu'+e^xu''-2\sin x=-2\sin x$

∴ $e^x(2u'+u'')=0$

∴ $u''+2u'=0$ $t^2+2t=0$ $t=0,-2$

∴ $u=C_1+C_2e^{-2x}$

∴ $y=C_1e^x+C_2e^{-x}+\sin x$ $y'=C_1e^x-C_2e^{-x}+\cos x$

$C_1+C_2=0$ $C_1-C_2+1=2$

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3.12 $\quad 2C_1 = 1 \quad C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{2}$

$$\therefore y = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} + \sin x$$

3.13 $\quad y'' - 4y' + y = e^{-x}\log x$

(1) $y = e^{-x}z$ とおくと $\quad y' = -e^{-x}z + e^{-x}z', \quad y'' = e^{-x}z - 2e^{-x}z' + e^{-x}z''$

$$e^{-x}z'' - 2e^{-x}z' + e^{-x}z + 4e^{-x}z - 4e^{-x}z' + e^{-x}z = e^{-x}\log x$$

$$z'' - 6z' + 6z = \log x$$

(2) $(D^2 - 6D + 6)z = \log x$

$t^2 - 6t + 6 = 0 \qquad t = 3 \pm \sqrt{3}$

$$\frac{1}{(D-3+\sqrt{3})(D-3-\sqrt{3})}\log x = \frac{1}{2\sqrt{3}}\left(\frac{1}{D-3-\sqrt{3}} - \frac{1}{D-3+\sqrt{3}}\right)\log x$$

$$= \frac{1}{2\sqrt{3}}e^{(3+\sqrt{3})x}\frac{1}{D}e^{(-3-\sqrt{3})x}\log x - \frac{1}{2\sqrt{3}}e^{(3-\sqrt{3})x}\frac{1}{D}e^{(-3+\sqrt{3})x}\log x$$

$$z = e^{(3+\sqrt{3})x}\left\{C_1 + \frac{1}{2\sqrt{3}}\int e^{-(3+\sqrt{3})x}\log x\,dx\right\}$$
$$+ e^{(3-\sqrt{3})x}\left\{C_2 - \frac{1}{2\sqrt{3}}\int e^{-(3-\sqrt{3})x}\log x\,dx\right\}$$

(3) $y = e^{(2+\sqrt{3})x}\left\{C_1 + \frac{1}{2\sqrt{3}}\int e^{-(3+\sqrt{3})x}\log x\,dx\right\} + e^{(2-\sqrt{3})x}\left\{C_2 - \frac{1}{2\sqrt{3}}\int e^{-(3-\sqrt{3})x}\log x\,dx\right\}$

P.72

§4. 2階微分方程式

4.1 $(1-x^2)y''+xy'=ax$

$y''+\frac{x}{1-x^2}y'=\frac{ax}{1-x^2}$ $\quad\int\frac{x}{1-x^2}dx=-\frac{1}{2}\log|1-x^2|$

$x^2<1$ のとき

$y'=\sqrt{1-x^2}\left\{\int a\frac{x}{\sqrt{1-x^2}}\frac{1}{(1-x^2)}dx+C_1\right\}$

$=\sqrt{1-x^2}\left\{a\frac{1}{\sqrt{1-x^2}}+C\right\}=a+\frac{C_1}{\sqrt{1-x^2}}$

$y=ax+C_1\sin^{-1}x+C_2$

$x^2>1$ のとき

$y'=\sqrt{x^2-1}\left\{\int\frac{-ax}{(\sqrt{x^2-1})^3}dx+C_1\right\}$

$=\sqrt{x^2-1}\left\{\frac{a}{\sqrt{x^2-1}}+C_1\right\}=a+\frac{C_1}{\sqrt{x^2-1}}$

$y=ax+C_1\{x\sqrt{x^2-1}-\log|x+\sqrt{x^2-1}|\}+C_2$

4.2 $x^2y''+Pxy'+Qy=0$ において $x=e^t$ とおく $t=\log x$

$y'=\frac{dy}{dt}\frac{dt}{dx}=\frac{1}{x}\frac{dy}{dt}$ $\quad y''=\frac{d}{dx}y'=\frac{-1}{x^2}\frac{dy}{dt}+\frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx}$

$=-\frac{1}{x^2}\frac{dy}{dt}+\frac{1}{x^2}\frac{d^2y}{dt^2}$

$\therefore\ \frac{d^2y}{dt^2}+(P-1)\frac{dy}{dt}+Qy=0$

(1) $x^2y''+3xy'-3y=3\log x-2$ $\quad x=e^t$ とおいて

$\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=3t-2$

$(D^2+2D-3)y=3t-2$ $\quad -\frac{1}{3}\frac{1}{1-\frac{2}{3}D-\frac{1}{3}D^2}(3t-2)$

$\lambda^2+2\lambda-3=0$ $\quad =-\frac{1}{3}(1+\frac{2}{3}D)(3t-2)=-\frac{1}{3}(3t-2+2)$

$(\lambda-1)(\lambda+3)=0$ $\quad =-t$

$\lambda=1,\ -3$

$y=C_1e^t+C_2e^{-3t}-t$

$y=C_1x+\frac{C_2}{x^3}-\log x$

(2) $x^2y''+xy'+y=0$ $\quad x=e^t$ とおいて

$\frac{d^2}{dt^2}y+y=0$ $\quad (D^2+1)y=0$ $\quad \lambda=\pm i$

$y=C_1\cos t+C_2\sin t$

$y=C_1\cos(\log x)+C_2\sin(\log x)$

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4.2 (3) $x^2y''+xy'+y=\log x$ $\quad x=e^t$ とおいて

$\frac{d^2}{dt^2}y+y=t \quad (D^2+1)y=t \quad \frac{1}{D^2+1}t=t$

$\lambda=\pm i$

$y=c_1\cos t+c_2\sin t+t$

$\therefore y=c_1\cos(\log x)+c_2\sin(\log x)+\log x$

(4) $x^2y''+3xy'+y=x$ $\quad x=e^t$ とおいて

$\frac{d^2}{dt^2}y+2\frac{d}{dt}y+y=e^t \quad (D^2+2D+1)y=e^t$

$\frac{1}{(D+1)^2}e^t=\frac{1}{4}e^t$

$(\lambda+1)^2=0 \quad \lambda=-1$

$\therefore y=(c_1+c_2t)e^{-t}+\frac{1}{4}e^t$

$y=\frac{c_1}{x}+\frac{c_2}{x}\log x+\frac{1}{4}x$

(5) $x^2y''+xy'+y=x$ $\quad x=e^t$ とおいて

$\frac{d^2}{dt^2}y+y=e^t \quad (D^2+1)y=e^t, \quad \frac{1}{D^2+1}e^t=\frac{1}{2}e^t$

$\lambda^2=-1 \quad \lambda=\pm i$

$y=c_1\cos t+c_2\sin t+\frac{1}{2}e^t$

$y=c_1\cos(\log x)+c_2\sin(\log x)+\frac{1}{2}x$

(6) $xy''+y'=0$ $\quad x=e^t$ とおく

$\frac{d^2}{dt^2}y=0 \quad D^2y=0$

$y=c_1+c_2t$

$y=c_1+c_2\log x$

4.3 (1) $yy''+(y')^2=a^2, \quad y(0)=y_0, \quad y'(0)=0$

$(yy')'=yy''+(y')^2$ より

$(yy')'=a^2 \quad yy'=a^2x+c_1$

$\frac{1}{2}y^2=\frac{1}{2}a^2x^2+c_1x+c_2 \quad y(0)=y_0, \quad y'(0)=0$ より

$\frac{1}{2}y_0^2=c_2 \quad c_1=0$

$\therefore y^2=a^2x^2+y_0^2$

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(2) $yy''+2(y')^2-2yy'=0 \qquad (yy')'=y'^2+yy''$

$(yy')'+(y')^2-2yy'=0 \qquad \frac{(yy')'}{yy'}+\frac{y'}{y}=2 \qquad \log|yy'|+\log|y|=2x+c$

$y^2y'=c_1e^{2x}$

$\therefore y^3=c_1e^{2x}+c_2$

(3) $yy''+(y')^2=0 \qquad (y'y)'=0 \qquad y'y=c_1 \qquad \frac{1}{2}y^2=c_1x+c_2$

$\therefore y^2=ax+b$

4.4 $x^2y''+xy'-4y=0 \qquad (x>0)$

(1) $y=x^{-2}$ を代入すると $y'=-2x^{-3} \quad y''=6x^{-4}$

$6x^{-2}-2x^{-2}-4x^{-2}=(6-6)x^{-2}=0$

$\therefore y=x^{-2}$ は解

(2) $g(x)=4x^{-2}\int x^4e^{-\log x}dx=4x^{-2}\int x^3dx=4x^{-2}\frac{x^4}{4}=x^2$

$y=g(x)=x^2$ を代入して

$2x^2+2x^2-4x^2=0 \qquad \therefore g(x)$ は解

(3) $c_1x^{-2}+c_2x^2=0$ は $c_1=c_2=0$ のときに限る

$\therefore x^{-2}, x^2$ は一次独立

$\therefore$ 一般解は $y=c_1x^{-2}+c_2x^2$

4.5 $x^2(x+1)y''-2x^2y'+2(x-1)y=0$

(1) $y=x^n$

$n(n-1)x^n(x+1)-2nx^{n+1}+2(x-1)x^n=0$

$\{n(n-1)-2n+2\}x^{n+1}+\{n(n-1)-2\}x^n=0$

$(n-2)(n-1)=0 \qquad (n-2)(n+1)=0 \qquad \therefore n=2$

(2) $y=x^2u \qquad y'=2xu+x^2u' \qquad y''=x^2u''+4xu'+2u$

$(x^3+x^2)\{x^2u''+4xu'+2u\}-2x^2(2xu+x^2u')+(2x-2)x^2u=0$

$(x^5+x^4)u''+(4x^4+4x^3-2x^4)u'+(2x^3+2x^2-4x^3+2x^3-2x^2)u=0$

$x^4(x+1)u''+x^3(2x+4)u'=0 \qquad x(x+1)u''+2(x+2)u'=0$

(3) $\frac{u''}{u'}=-\frac{2x+4}{x(x+1)}=(\frac{2}{x+1}-\frac{4}{x})$

$\log u'=2\log|x+1|-4\log|x|+c=\log\frac{(x+1)^2}{x^4}+c$

$u'=c(\frac{1}{x^2}+\frac{2}{x^3}+\frac{1}{x^4}), \quad u=c(-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{3x^3})+c_2$

$\therefore y=c_2x^2-c_1(x+1+\frac{1}{3x})$

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4.6 $y'' + p(x)y' + Q(x)y = 0$ $\quad t = \int_0^x [e^{-\int p dx}] dx$

$y' = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt} \cdot e^{-\int p dx}$, $\cdot\ y'' = -p e^{-\int p dx} \frac{dy}{dt} + \frac{d^2y}{dt^2} e^{-2\int p dx}$

$\therefore\ \frac{d^2 y}{dt^2} + Q e^{2\int p dx} y = 0$

$y'' + \tan x\, y' + \cos^2 x\, y = 0$ $\quad p = \tan x$ $\quad \int p dx = -\log|\cos x|$

$t = \int_0^x \cos x\, dx = \sin x$ とおくと

$\frac{d^2}{dt^2}y + \cos^2 x \frac{1}{\cos^2 x} y = 0$ $\quad \therefore\ \frac{d^2}{dt^2}y + y = 0$ $\quad (D^2+1)y = 0$

$\lambda^2 + 1 = 0$ $\quad \lambda = \pm i$

$\therefore\ y = C_1 \cos t + C_2 \sin t = C_1 \cos(\sin x) + C_2 \sin(\sin x)$

4.7 $\frac{d^2 f}{dx^2} + (2x - a)\frac{df}{dx} + (x^2 - ax + 1) f = 0$

(1) $f(x) = e^{-\frac{1}{2}x^2} g(x)$ とおく $\quad$ ($e - \frac{1}{2}x^2$ は プリントミス)

$\frac{df}{dx} = -x e^{-\frac{1}{2}x^2} g(x) + e^{-\frac{1}{2}x^2} g'(x)$

$\frac{d^2 f}{dx^2} = (x^2 - 1) e^{-\frac{1}{2}x^2} g(x) - 2x e^{-\frac{1}{2}x^2} g'(x) + e^{-\frac{1}{2}x^2} g''(x)$

$g''(x) - 2x g'(x) + (x^2-1) g(x) + (2x - a) g' - (2x^2 - ax) g(x) + (x^2 - ax + 1) g(x) = 0$

$g''(x) - a g'(x) + (x^2 - 1 - 2x^2 + ax + x^2 - ax + 1) g(x) = 0$

$g''(x) - a g'(x) = 0$

(2) $\frac{dh}{dx} - a h = 0$ $\quad \log h = ax + c$ $\quad h = C e^{ax}$

$\therefore\ g'(x) = C e^{ax}$ $\quad g(x) = \frac{C}{a} e^{ax} + c = C_1 e^{ax} + C_2$

$\therefore\ f(x) = e^{-\frac{1}{2}x^2} (C_1 e^{ax} + C_2)$

(3) $f'(x) = e^{-\frac{1}{2}x^2} (-C_1 x e^{ax} - C_2 x + C_1 a e^{ax})$

$f(0) = C_1 + C_2 = 1$ $\quad f'(0) = C_1 a = a$ $\quad C_1 = 1,\ C_2 = 0$

$\therefore\ f(x) = e^{-\frac{1}{2}x^2 + ax} = e^{-\frac{1}{2}(x-a)^2 + \frac{a^2}{2}}$

$\therefore$ $f(x)$ は $x = a$ で最大値 $e^{\frac{a^2}{2}}$ をとる.

4.8 $2x^2 \frac{dy}{dx} - x^2 y^2 + 2xy + 1 = 0$

(1) $u = xy$ $\quad u' = y + xy'$ $\quad x^2 y' = xu' - xy = xu' - u$

$2xu' - 2u - u^2 + 2u + 1 = 0$ $\quad 2xu' = u^2 - 1$

(2) $\left(\frac{1}{u-1} - \frac{1}{u+1}\right) du = \frac{1}{x} dx$ $\quad \log \frac{u-1}{u+1} = \log|x| + C_1$

$xy - 1 = Cx(xy+1)$ $\quad xy = \frac{1+Cx}{1-Cx}$ $\quad y = (1+Cx)/x(1-Cx)$

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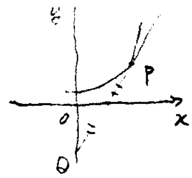
§5. 微分方程式の応用（図形）

5.1 (1) P(x, y) における接線



$Y-y=f'(x)(X-x)$ 　　Q： $X=0$ 　$Y=y-f'(x)x$

$Q(0,\ y-f'(x)\cdot x)$ 　PQの中点

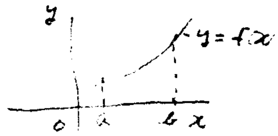
$\left(\dfrac{x}{2},\ \dfrac{2y-f'(x)\cdot x}{2}\right)$ これがx軸上にある

$\therefore\ 2y=f'(x)\cdot x$ 　　$\therefore\ x\cdot y'=2y$

(2) $\dfrac{y'}{y}=\dfrac{2}{x}$ 　$\therefore\ \log|y|=2\log|x|+C_1$ 　$\therefore\ y=Ax^2$

$y(2)=1$ 　$1=4A$ 　$\therefore\ A=\dfrac{1}{4}$. 　$y=\dfrac{1}{4}x^2$

5.2 面積 $S=\displaystyle\int_a^x f(x)\,dx=S(x)$



弧の長さ $l=\displaystyle\int_a^x\sqrt{1+(f'(x))^2}\,dx=l(x)$

$S(x)=k\,l(x)$ 　　$S'(x)=k\,l'(x)$ 　$f(x)=y$ とおいて

$y=k\sqrt{1+y'^2}$ 　　$k^2y'^2=y^2-k^2$ 　. $\underline{k\,y'=\sqrt{y^2-k^2}}$

$\dfrac{y'}{\sqrt{y^2-k^2}}=\dfrac{1}{k}$ 　　$\log|y+\sqrt{y^2-k^2}|=\dfrac{x+c}{k}$

$y+\sqrt{y^2-k^2}=Ae^{\frac{x}{k}}$ 　　$y(0)=k$ より 　$k=A$ 　$y+\sqrt{y^2-k^2}=ke^{\frac{x}{k}}$

$(y^2-k^2)=(ke^{\frac{x}{k}}-y)^2=k^2e^{\frac{2x}{k}}-2yke^{\frac{x}{k}}+y^2$

$\therefore\ 2yke^{\frac{x}{k}}=k^2e^{\frac{2x}{k}}+k^2$ 　　$y=\dfrac{k}{2}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)$

$y=k\cosh\left(\dfrac{x}{k}\right)$

5.3 P(x, y) における接線 　$Y-y=y'(X-x)$ 　y軸との交点

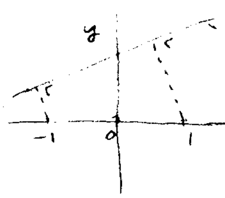
$Y=y-y'x$ 　$(0,\ y-y'x)$ 　$\therefore\ x=y-y'x$

$y'-\dfrac{1}{x}y=-1$ 　　$\displaystyle\int\frac{1}{x}\,dx=\log|x|$

$y=x\left(\displaystyle\int\frac{-1}{x}\,dx+C\right)=x(-\log|x|+C)$

$\therefore\ y=-x\log|x|+Cx$

5.4 (1)



$y=ax+b$ 　　$-ax+y=b$

(−1, 0) からの垂線の長さ $\dfrac{|a-b|}{\sqrt{1+a^2}}$

(1, 0) 　〃 　$\dfrac{|a+b|}{\sqrt{1+a^2}}$

$b^2-a^2=\pm k(1+a^2)$

$\therefore\ b^2=a^2\pm k(1+a^2)$ 　　$b=\pm\sqrt{a^2\pm k(1+a^2)}$

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5.4 (2) $k=1$ のとき $b = \pm\sqrt{2a^2+1}$

$\therefore\ y = ax \pm \sqrt{2a^2+1}$, $\quad \dfrac{\partial}{\partial a}\quad x \pm \dfrac{2a}{\sqrt{2a^2+1}} = 0$

$x = \mp \dfrac{2a}{\sqrt{2a^2+1}} \qquad y = \dfrac{\pm 1}{\sqrt{2a^2+1}}$

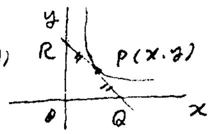
$\dfrac{x^2}{2} + y^2 = 1 \qquad \therefore\ x^2 + 2y^2 = 2$

5.5 $P(x, y)$ における法線 $Y - y = \dfrac{-1}{y'}(X - x)$ これが点 (a, b) を通る

$\therefore\ b - y = -\dfrac{1}{y'}(a - x) \qquad (y-b)y' + (x-a) = 0$

$\therefore\ (y-b)^2 + (x-a)^2 = C^2$

5.6



接線 $Y - y = y'(X - x)$

(1) Q. $Y = 0$. $X = x - \dfrac{y}{y'}$ $\qquad \left(x - \dfrac{y}{y'},\ 0\right)$

R $\quad X = 0 \quad Y = y - y'x \qquad (0,\ y - y'x)$

QRの中点 $\left(\dfrac{1}{2}\left(x - \dfrac{y}{y'}\right),\ \dfrac{1}{2}(y - y'x)\right)$ これがPと一致する

$x = \dfrac{1}{2}\left(x - \dfrac{y}{y'}\right), \quad y = \dfrac{1}{2}(y - y'x)$

$x = -\dfrac{y}{y'} \qquad\qquad y = -y'x$

$\dfrac{y'}{y} + \dfrac{1}{x} = 0 \qquad\qquad \dfrac{y'}{y} + \dfrac{1}{x} = 0$

(2) $\log|yx| = C \qquad \log|xy| = C_1 \qquad \therefore\ xy = a$

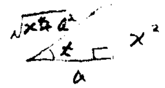
これが $(\sqrt{3}, \sqrt{7})$ を通る $\quad a = \sqrt{21} \qquad \therefore\ xy = \sqrt{21}$

(3) $\sqrt{2} \leqq x \leqq \sqrt{10} \qquad y = \dfrac{\sqrt{21}}{x}$

$y' = -\dfrac{\sqrt{21}}{x^2} \qquad \sqrt{1 + y'^2} = \sqrt{1 + \dfrac{21}{x^4}} = \dfrac{1}{x^2}\sqrt{x^4 + 21}$

$\therefore\ S = 2\pi \displaystyle\int_{\sqrt{2}}^{\sqrt{10}} \frac{\sqrt{21}}{x^3}\sqrt{x^4+21}\,dx$

$\displaystyle\int \frac{1}{x^3}\sqrt{x^4 + a^2}\,dx \qquad\qquad \frac{x^2}{a} = \tan t$ とおく



$x\,dx = \dfrac{a}{2}\sec^2 t\,dt$

$= \displaystyle\int \frac{x}{x^4}\sqrt{x^4 + a^2}\,dx$

$= \displaystyle\int \frac{1}{a^2 \tan^2 t}\, a \sec t\, \frac{a}{2}\sec^2 t\,dt = \frac{1}{2}\int \frac{1}{\sin^2 t \cos t}\,dt$

$= \dfrac{1}{2}\displaystyle\int \frac{\cos t}{\sin^2 t(1 - \sin^2 t)}\,dt = \frac{1}{2}\int \left\{\frac{1}{\sin^2 t} + \frac{1}{2(1 - \sin t)} + \frac{1}{2(1 + \sin t)}\right\}\cos t\,dt$

$= \dfrac{1}{2}\left(-\dfrac{1}{\sin t} + \dfrac{1}{2}\log\left|\dfrac{1 + \sin t}{1 - \sin t}\right|\right) = -\dfrac{\sqrt{x^4 + a^2}}{2x^2} + \dfrac{1}{4}\log\left|\dfrac{\sqrt{x^4 + a^2} + x^2}{\sqrt{x^4 + a^2} - x^2}\right|$

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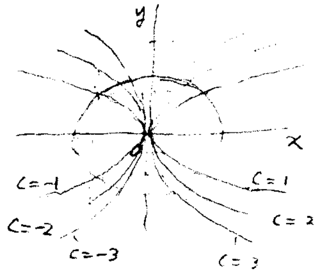
$$S = 2\sqrt{21}\pi \int_{\sqrt{2}}^{\sqrt{10}} \frac{1}{x^3}\sqrt{x^4+21}\,dx$$

$$= 2\sqrt{21}\pi \left[-\frac{\sqrt{x^4+21}}{2x^2} + \frac{1}{4}\log\left|\frac{\sqrt{x^4+21}+x^2}{\sqrt{x^4+21}-x^2}\right| \right]_{\sqrt{2}}^{\sqrt{10}}$$

$$= 2\sqrt{21}\pi \left\{ \frac{5}{4} - \frac{11}{20} + \frac{1}{4}\log \frac{11+10}{11-10} \Big/ \frac{5+2}{5-2} \right\}$$

$$= 2\sqrt{21}\pi \left(\frac{7}{10} + \frac{1}{4}\log 21 \cdot \frac{3}{7} \right) = \sqrt{21}\pi \left(\frac{7}{5} + \log 3 \right)$$

5.7 (1) $y^2 = 4cx$ $\quad c = \pm 1 \quad c = \pm 2 \quad c = \pm 3$



(2) $2yy' = 4c$

$y^2 = 2yy'x \qquad y = 2y'x$

(3) (2)より $y' = \dfrac{y}{2x} \qquad y' = -\dfrac{2x}{y}$

$\therefore\ yy' = -2x \qquad \therefore\ \dfrac{y^2}{2} + x^2 = a^2$

5.8 $y^2 = cx \qquad 2yy' = c \qquad y^2 = 2yy'x \qquad y = 2y'x$

$y' = \dfrac{y}{2x} \qquad y' = -\dfrac{2x}{y} \qquad \dfrac{y^2}{2} + x^2 = a^2$

5.9 接線 $Y - y = y'(X - x)$ $\quad$ Q: $Y = 0$ $\quad X = x - \dfrac{y}{y'}$ $\quad Q\left(x - \dfrac{y}{y'}, 0\right)$

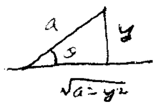
$$\overline{PQ} = \sqrt{\left\{x - \left(x - \frac{y}{y'}\right)\right\}^2 + y^2} = \sqrt{\frac{y^2}{y'^2} + y^2} = a$$

$y^2(1 + y'^2) = a^2 y'^2 \qquad y'^2(a^2 - y^2) = y^2, \quad y' = \dfrac{y}{\sqrt{a^2 - y^2}}$

$\displaystyle\int \frac{\sqrt{a^2-y^2}}{y}\,dy = \int dx \qquad \int \frac{\sqrt{a^2-y^2}}{y}\,dy$

$y = a\sin t$

$dy = a\cos t\,dt$



$$= \int \frac{a^2\cos^2 t}{a\sin t}\,dt$$

$$= \int a\,\frac{\cos^2 t}{1-\cos^2 t}\sin t\,dt = a\int \left(-1 + \frac{1}{1-\cos^2 t}\right)\sin t\,dt$$

$$= a\int \left(-1 + \frac{1}{2(1-\cos t)} + \frac{1}{2(1+\cos t)}\right)\sin t\,dt$$

$$= a\cos t + \frac{a}{2}\log\left|\frac{1-\cos t}{1+\cos t}\right| = \sqrt{a^2-y^2} + \frac{a}{2}\log\left|\frac{\sqrt{a^2-y^2}-a}{\sqrt{a^2-y^2}+a}\right|$$

$$= \sqrt{a^2-y^2} + a\log\left|\frac{a-\sqrt{a^2-y^2}}{y}\right|$$

$$\therefore\ \sqrt{a^2-y^2} + a\log\left|\frac{a-\sqrt{a^2-y^2}}{y}\right| = x + C$$

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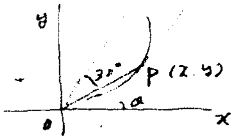
5.10



(1) $y' = \tan\alpha$, $\alpha = \tan^{-1} y'$

$\alpha = \tan^{-1}\frac{y}{x} + \frac{\pi}{6}$

∴ $\tan^{-1} y' - \tan^{-1}\frac{y}{x} = \frac{\pi}{6}$

$\tan(\tan^{-1} y' - \tan^{-1}\frac{y}{x}) = \frac{1}{\sqrt{3}}$ $\qquad \frac{y' - \frac{y}{x}}{1 + y'\cdot\frac{y}{x}} = \frac{1}{\sqrt{3}}$

$xy' - y = \frac{1}{\sqrt{3}}(x + yy')$ $\qquad (\sqrt{3}x - y)y' = x + \sqrt{3}y$

(2) $x = r\cos\theta$, $y = r\sin\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$

$(\sqrt{3}\cos\theta - \sin\theta)r(r'\sin\theta + r\cos\theta) = r(\cos\theta + \sqrt{3}\sin\theta)(r'\cos\theta - r\sin\theta)$

$r'(\sqrt{3}\cos\theta\sin\theta - \sin^2\theta - \cos^2\theta - \sqrt{3}\sin\theta\cos\theta)$

$= r(-\sin\theta\cos\theta - \sqrt{3}\sin^2\theta + \sin\theta\cos\theta - \sqrt{3}\cos^2\theta)$

$r' = \sqrt{3}r$

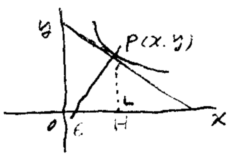
(3) $\frac{r'}{r} = \sqrt{3}$ $\qquad \log|r| = \sqrt{3}\theta + C$, $r = ae^{\sqrt{3}\theta}$

$(0, 1)$ $\quad r = 1 \quad \theta = \frac{\pi}{2}$ $\qquad 1 = ae^{\frac{\sqrt{3}}{2}\pi}$ $\qquad a = e^{-\frac{\sqrt{3}}{2}\pi}$

∴ $r = e^{\sqrt{3}(\theta - \frac{\pi}{2})}$ $\qquad r^2 = e^{\sqrt{3}(2\theta - \pi)}$

∴ $x^2 + y^2 = \exp(2\sqrt{3}\tan^{-1}\frac{y}{x} - \sqrt{3}\pi)$

5.11 (1)



$QH = a$

法線 $Y - y = -\frac{1}{y'}(X - x)$

Q : $Y = 0$ $\qquad X = x + yy'$

$Q(x + yy', 0)$ $\qquad H(x, 0)$

∴ $yy' = a$ $\qquad \frac{1}{2}y^2 = ax + C$

∴ $y^2 = 2ax + b$.

(2) $(1, 0)$を通る $\qquad 0 = 2a + b$ $\qquad b = -2a$

∴ $y^2 = 2a(x - 1)$

P.74

5.12 $x^2+y^2=cx$

(1) $2x+2yy'=c$ $\quad x^2+y^2=2x^2+2xyy'$

$y'=\dfrac{y^2-x^2}{2xy}$

(2) $y'=\dfrac{2xy}{x^2-y^2}$ $\quad y=ux$ とおくと

$u'x+u=\dfrac{2u}{1-u^2}$ $\quad u'x=\dfrac{u+u^3}{1-u^2}$ $\quad \dfrac{1-u^2}{u(1+u^2)}du=\dfrac{1}{x}dx$

$\left(\dfrac{1}{u}-\dfrac{2u}{1+u^2}\right)du=\dfrac{1}{x}dx$ $\quad \log\left|\dfrac{u}{1+u^2}\right|=\log|x|+c$

$u=ax(1+u^2)$ $\quad y=a(x^2+y^2)$

$\therefore\ x^2+y^2=cy.$

5.13 (1) $y=f(x)$ とする。 $y'=f'(x)$ $\quad f'(x_0)=a$

$y-y_0=a(x-x_0)$ $\quad y=a(x-x_0)+y_0$

(2) 法線

$Y-y=\dfrac{-1}{y'}(X-x)$ これが $(1,2)$ を通る

$2-y=\dfrac{-1}{y'}(1-x)$ $\quad \underline{(y-2)y'=-(x-1)}$

$(y-2)^2+(x-1)^2=c^2$

(3) $(y-2)^2+(x-1)^2=c^2$ が $y=3x$ に接する。

$(3x-2)^2+(x-1)^2=c^2$

$10x^2-14x+5-c^2=0$

$49-10(5-c^2)=0$ $\quad -1+10c^2=0$ $\quad c^2=\dfrac{1}{10}$

$(y-2)^2+(x-1)^2=\dfrac{1}{10}$

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§.6 微分方程式の応用（現象）

6.1 $V=\frac{4}{3}\pi R^3$

$\frac{dV}{dt}=4\pi R^2\frac{dR}{dt}=-kR \qquad \frac{dR}{dt}=-\frac{k}{4\pi R} \qquad R^2=-\frac{k}{2\pi}t+C$

$t=0$ のとき $R=R_0 \qquad R_0^2=C \quad \therefore\ R^2=-\frac{k}{2\pi}t+R_0^2$

$R=0$ のとき $T=t \qquad 0=-\frac{k}{2\pi}T+R_0^2 \qquad k=\frac{2\pi R_0^2}{T}$

$\therefore\ R^2=-\frac{R_0^2}{T}t+R_0^2 \qquad R=R_0\sqrt{1-\frac{t}{T}}$

6.2 $\frac{y'}{y}=k(a-y) \qquad \frac{1}{y(a-y)}y'=k \qquad \left(\frac{1}{y}+\frac{1}{a-y}\right)y'=ak$

$\log\left|\frac{y}{a-y}\right|=akx+C_1 \qquad \therefore\ y=Ce^{akx}(a-y)$

$x=0$ のとき $y=N \qquad N=C(a-N) \quad \therefore\ C=\frac{N}{a-N}$

$y=\frac{N}{a-N}e^{akt}(a-y) \qquad \{(a-N)+Ne^{akt}\}y=Nae^{akt}$

$$\therefore\ y=\frac{Nae^{akt}}{a-N+Ne^{akt}}$$

6.3 $mx''=-kx'-Rx \qquad (mD^2+kD+R)x=0$

補助方程式 $m\lambda^2+k\lambda+R=0 \qquad \lambda=\frac{-k\pm\sqrt{k^2-4mR}}{2m}$

$k^2>4mR$ のとき $\alpha=\frac{-k+\sqrt{k^2-4mR}}{2m},\ \beta=\frac{-k-\sqrt{k^2-4mR}}{2m}$ とおけば

解は $x=C_1e^{\alpha t}+C_2e^{\beta t}$

$k^2=4mR$ のとき $x=(C_1+C_2t)e^{-\frac{k}{2m}t}$

$k^2<4mR$ のとき $x=e^{-\frac{k}{2m}t}\left(C_1\cos\frac{\sqrt{4mR-k^2}}{2m}t+C_2\sin\frac{\sqrt{4mR-k^2}}{2m}t\right)$

$\therefore$ 減衰振動となる条件は $\frac{k}{2m}>0,\ k^2<4mR$

P 74　§8 級数による解法

7.1　$y''-y=0 \quad y=\sum_{n=0}^{\infty} a_n x^n$

(1) $\sum_{n=2} n(n-1)a_n x^{n-2} - \sum_{n=0} a_n x^n = 0$　第1項で $n=m+2$ とおくと

$\sum_{m=0}(m+2)(m+1)a_{m+2}x^m - \sum_{n=0} a_n x^n = 0$

$\therefore (n+2)(n+1)a_{n+2}-a_n=0 \quad a_{n+2}=\frac{1}{(n+2)(n+1)}a_n$

$\therefore a_{2n}=\frac{1}{(2n)!}a_0 \quad a_{2n-1}=\frac{1}{(2n-1)!}a_1$

(2) $y(0)=2,\ a_0=2,\ y'(0)=0 \quad a_1=0$

$a_{2n}=\frac{1}{(2n)!} \quad a_{2n-1}=0$

7.2　$x(x-1)y''+\{(\alpha+\beta+1)x-\gamma\}y'+\alpha\beta y=0$

(1) $y=\sum_{n=0} c_n x^n \quad y'=\sum_{n=1} n c_n x^{n-1} \quad y''=\sum_{n=2} n(n-1)c_n x^{n-2}$

$x^2y''+(\alpha+\beta+1)y'+\alpha\beta y - xy'' - \gamma y'=0$

$\sum_{n=2} n(n-1)c_n x^n + \sum_{n=1} n(\alpha+\beta+1)c_n x^n + \sum_{n=0}\alpha\beta c_n x^n - \sum_{n=2} n(n-1)c_n x^{n-1} - \sum_{n=1}\gamma n c_n x^{n-1}=0$

$\alpha\beta c_0-\gamma c_1=0 \quad (\alpha\beta+\alpha+\beta+1)c_1-2(1+\gamma)c_2=0$

$\{n(n-1)+n(\alpha+\beta+1)+\alpha\beta\}c_n - \{n(n+1)+(n+1)\gamma\}c_{n+1}=0$

$\therefore c_1=\frac{\alpha\beta}{\gamma}c_0 \quad c_2=\frac{(\alpha+1)(\beta+1)}{2(\gamma+1)}c_1 \quad c_{n+1}=\frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)}$

$\therefore c_n=\prod_{k=1}^{n}\frac{(\alpha+k)(\beta+k)}{k(\gamma+k)}c_0$

(2) $\lim_{n\to\infty}\frac{c_n}{c_{n+1}}=\lim_{n\to\infty}\frac{(n+1)(\gamma+n+1)}{(\alpha+n+1)(\beta+n+1)}=1$　収束半径 1

7.3　$y'=y \quad y(0)=1$

(1) $y=\sum_{n=0} a_n x^n \quad y'=\sum_{n=1} n a_n x^{n-1}$

$y'-y=0$ より　$\sum_{n=0}\{a_n-(n+1)a_{n+1}\}x^n=0 \quad \therefore a_{n+1}=\frac{1}{n+1}a_n$

$y(0)=1$ より $a_0=1 \quad \therefore y=\sum_{n=0}\frac{1}{n!}x^n$　　$0!=1$ とする。

(2) $e^x=\sum_{n=0}\frac{1}{n!}x^n$

$y'=y,\ y(0)=e^a$ の解は $\sum_{n=0}\frac{e^a}{n!}x^n=e^a e^x$

また e^{a+x} は $y'=y,\ y(0)=e^a$ の解である

$\therefore e^a e^x=e^{a+x} \quad \therefore e^a e^b=e^{a+b}$

P. 74, 75

§8 連立微分方程式

8.1 (1) $\begin{cases} x'-3y'=-2y \\ x'-5y'=-4x \end{cases}$ $\begin{cases} Dx-(3D-2)y=0 \quad \cdots ① \\ (D+4)x-5Dy=0 \quad \cdots ② \end{cases}$ $D=\frac{d}{dt}$

$5D\cdot① - (3D-2)\cdot②$ より

$\{5D^2-(D+4)(3D-2)\}x=0 \qquad (2D^2-10D+8)x=0$

特性方程式 $t^2-5t+4=0 \quad t=1,4$

$\therefore x=C_1e^t+C_2e^{4t} \qquad 5Dy=(D+4)x=5C_1e^t+8C_2e^{4t}$

$\therefore y=\frac{1}{5}\frac{1}{D}(5C_1e^t+8C_2e^{4t})$

$y=C_1e^t+\frac{2}{5}C_2e^{4t}$

(2) $x'=x+2y$
$y'=-x+4y$

$\begin{pmatrix}1&2\\-1&4\end{pmatrix}$ の固有値 $\begin{vmatrix}1-\lambda&2\\-1&4-\lambda\end{vmatrix}=0$

$\lambda^2-5\lambda+6=0 \quad (\lambda-2)(\lambda-3)=0 \quad \lambda=2,3$

$\lambda=2$ の固有ベクトル $\begin{pmatrix}-1&2\\-1&2\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

$\lambda=3$ の固有ベクトル $\begin{pmatrix}-2&2\\-1&1\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

$\begin{cases} x=2C_1e^{2t}+C_2e^{3t} \\ y=C_1e^{2t}+C_2e^{3t} \end{cases}$

(3) $\begin{cases} y'=6y+2z \\ z'=-y+4z \end{cases}$ $\begin{cases} (6-D)y+2z=0 \\ -y+(4-D)z=0 \end{cases}$ $y=z=0$ 以外に解をもつのは

$\begin{vmatrix}6-D&2\\-1&4-D\end{vmatrix}=0 \qquad D^2-10D+26=0 \qquad \lambda^2-10\lambda+26=0 \quad \lambda=5\pm i$

$\therefore y=e^{5x}(C_1\cos x+C_2\sin x) \qquad y'=e^{5x}\{(5C_1+C_2)\cos x+(5C_2-C_1)\sin x\}$

$z=\frac{1}{2}(y'-6y)=\frac{1}{2}e^{5x}\{(C_2-C_1)\cos x-(C_2+C_1)\sin x\}$

$\therefore y=2e^{5x}(A\cos x+B\sin x)$

$z=e^{5x}\{(B-A)\cos x-(A+B)\sin x\}$

8.2 (1) $\begin{cases} x'=x+2y \\ y'=2x+y \end{cases}$ $\begin{cases} x(0)=1 \\ y(0)=0 \end{cases}$ $\begin{cases} (1-D)x+2y=0 \\ 2x+(1-D)y=0 \end{cases}$ $\begin{vmatrix}1-\lambda&2\\2&1-\lambda\end{vmatrix}=0$

$\lambda^2-2\lambda-3=0 \quad (\lambda-3)(\lambda+1)=0 \quad \lambda=3,\ -1$

固有値 $\lambda=3$ の固有ベクトル $\begin{pmatrix}-2&2\\2&-2\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

固有値 $\lambda=-1$ の固有ベクトル $\begin{pmatrix}2&2\\2&2\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

$\therefore x=C_1e^{3t}+C_2e^{-t} \qquad \begin{cases} x(0)=1 \\ y(0)=1 \end{cases}$ より $\quad \begin{cases} 1=C_1+C_2 \\ 0=C_1-C_2 \end{cases} \quad C_1=C_2=\frac{1}{2}$

$y=C_1e^{3t}-C_2e^{-t}$

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$x = \frac{1}{2}(e^{3t} + e^{-t})$

$y = \frac{1}{2}(e^{3t} - e^{-t})$

8.3 $\begin{cases} x' = 3x - 5y \\ y' = 5x - 3y \end{cases}$ $\quad x(0) = 1 \quad y(0) = 1$

(1) $\begin{cases} u = x + y \\ v = -2x + 2y \end{cases}$ $\quad u' = x' + y' = 3x - 5y + 5x - 3y = 8x - 8y$

$= -4v$

$v' = -2x' + 2y' = -6x + 10y + 10x - 6y = 4x + 4y$

$= 4u$

$\therefore \begin{cases} u' = -4v \\ v' = 4u \end{cases}$

(2) (1)より $u'' + 4v' = 0 \quad u'' + 16u = 0 \quad \therefore \quad u = C_1 \cos 4t. \quad v = C_2 \sin 4t$

$x = \frac{1}{4}(2u - v) = \frac{1}{2} C_1 \cos 4t - \frac{1}{4} C_2 \sin 4t$

$y = \frac{1}{4}(2u + v) = \frac{1}{2} C_1 \cos 4t + \frac{1}{4} C_2 \sin 4t$

$\therefore \quad x = A \cos 4t - B \sin 4t \quad A = 1 \quad B = \frac{1}{2} \quad \therefore \begin{cases} x = \cos 4t - \frac{1}{2} \sin 4t \\ y = \cos 4t + \frac{1}{2} \sin 4t \end{cases}$

$y = A \cos 4t + B \sin 4t$

8.4 $a_0 = 1 \quad b_0 = 1. \quad a_n = 3a_{n-1} + b_{n-1} \quad b_n = 2a_{n-1} + 2b_{n-1}$

(1) $a_n - b_n = 3a_{n-1} + b_{n-1} - 2a_{n-1} - 2b_{n-1} = a_{n-1} - b_{n-1}$

$\therefore \quad a_n - b_n = a_{n-1} - b_{n-1} = \qquad = a_0 - b_0 = 0$

$\therefore \quad y'' - y' - 6y = 0$ 特性方程式 $t^2 - t - 6 = 0$

$(t-3)(t+2) = 0 \qquad t = 3, -2$

$y = C_1 e^{3t} + C_2 e^{-2t}$

(2) $y' = z$ とおくと $z' = z + 6y$

$\therefore \begin{cases} y' = z \\ z' = 6y + z \end{cases}$

(3) $t = 0$ のとき $y = 0.\ y' = 1$

(1)より $0 = C_1 + C_2 \qquad 1 = 3C_1 - 2C_2 \qquad C_1 = \frac{1}{5} \quad C_2 = -\frac{1}{5}$

$\therefore \quad y = \frac{1}{5}(e^{3t} - e^{-2t})$

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8.5 $f'(x)=g(x)$, $g'(x)=f(x)$ $y=f(x)$, $z=g(x)$ とおくと

(1) $\begin{cases} y'=z \\ z'=y \end{cases}$ $y''=y$ $\lambda^2-1=0$ $\lambda=\pm 1$

$\therefore y=c_1e^{x}+c_2e^{-x}$ $z=c_1e^{x}-c_2e^{-x}$

$f(0)=1$, $g(0)=0$ より $1=c_1+c_2$ $0=c_1-c_2$ $c_1=c_2=\frac{1}{2}$

$\therefore$ $f(x)=\frac{1}{2}(e^{x}+e^{-x})$

$g(x)=\frac{1}{2}(e^{x}-e^{-x})$

(2) $f^{(n)}(x)=\frac{1}{2}\{e^{x}+(-1)^n e^{-x}\}$

$g^{(n)}(x)=\frac{1}{2}\{e^{x}-(-1)^n e^{-x}\}$

(3) $f(x)\cdot g(y)+g(x)\cdot f(y)$

$=\frac{1}{4}(e^{x}+e^{-x})(e^{y}-e^{-y})+\frac{1}{4}(e^{y}+e^{-y})(e^{x}-e^{-x})$

$=\frac{1}{4}\{e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}+e^{x+y}-e^{-x+y}+e^{-y+x}-e^{-x-y}\}$

$=\frac{1}{2}(e^{x+y}-e^{-(x+y)})=g(x+y)$

$\therefore f(x)g(y)+f(y)g(x)=g(x+y)$

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§9 行列微分方程式

9.1 $A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$

(1) $\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$ $\quad \begin{cases} -1+3a=1 \\ -2+4a=a \end{cases}$ $\quad a = \frac{2}{3}$

(2) $x_2 = \begin{pmatrix} 1 \\ b \end{pmatrix}$ $\quad \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = 2\begin{pmatrix} 1 \\ b \end{pmatrix}$ $\quad \begin{matrix} -1+3b=2 \\ -2+4b=2b \end{matrix}$ $\quad b=1$

(3) $T = (x_1\ x_2) = \begin{pmatrix} 1 & 1 \\ \frac{2}{3} & 1 \end{pmatrix}$ $\quad T^{-1} = 3\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & 1 \end{pmatrix}$

$$T^{-1}AT = 3\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \frac{2}{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

(4) $\vec{y}'(x) = A\vec{y}(x)$, $\vec{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \vec{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ とおく

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Aの固有値は1, 2で固有ベクトル $\begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{2}{3} & 1 \end{pmatrix}\begin{pmatrix} e^x \\ e^{2x} \end{pmatrix}$$

$y_1 = c_1 e^x + c_2 e^{2x}$ $\quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y_2 = \frac{2}{3}c_1 e^x + c_2 e^{2x}$

$1 = c_1 + c_2$ $\quad -1 = \frac{1}{3}c_1$ $\quad c_1 = -3$ $\quad c_2 = 4$

$2 = \frac{2}{3}c_1 + c_2$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3e^x + 4e^{2x} \\ -2e^x + 4e^{2x} \end{pmatrix}$$

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9.2 $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x-2y \\ x+4y \end{pmatrix}$

(1) $\begin{pmatrix} x-2y \\ x+4y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \therefore A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

(2) $\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = 0 \quad \lambda^2 - 5\lambda + 6 = 0 \quad \lambda = 2, 3$

固有値$\lambda=2$の固有ベクトル $\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

固有値$\lambda=3$の固有ベクトル $\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(3) $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix} \quad P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad P^{-1} = -\sqrt{10}\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = P\begin{pmatrix} u' \\ v' \end{pmatrix}$

$\therefore P\begin{pmatrix} u' \\ v' \end{pmatrix} = AP\begin{pmatrix} u \\ v \end{pmatrix} \quad \therefore \begin{pmatrix} u' \\ v' \end{pmatrix} = P^{-1}AP\begin{pmatrix} u \\ v \end{pmatrix}$

(4) $\begin{pmatrix} u' \\ v' \end{pmatrix} = P^{-1}AP\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$u' = 2u \qquad u = c_1 e^{2t}$

$v' = 3v \qquad v = c_2 e^{3t}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} 2c_1 e^{2t} + c_2 e^{3t} \\ -c_1 e^{2t} - c_2 e^{3t} \end{pmatrix}$

P76 § 10 積分方程式

10.1 $f(x)=(ax+b)e^{\frac{x}{2}}$

$f(x)=e^{\frac{x}{2}}-1+\frac{1}{2}\int_0^x f(t)\,dt$

$=e^{\frac{x}{2}}-1+\frac{1}{2}\int_0^x(at+b)e^{\frac{t}{2}}dt$

$=e^{\frac{x}{2}}-1+\frac{1}{2}\left[2(at+b)e^{\frac{t}{2}}\right]_0^x-a\int_0^x e^{\frac{t}{2}}dt$

$=e^{\frac{x}{2}}-1+(ax+b)e^{\frac{x}{2}}-b-2ae^{\frac{x}{2}}+2a$

$=(1+ax+b-2a)e^{\frac{x}{2}}+2a-b-1$

$\therefore$ $ax+b-2a+1=ax+b$ $2a-b-1=0$

$\begin{cases}-2a+1=0\\2a-b-1=0\end{cases}$ $a=\frac{1}{2}$ $b=0$

10.2 $y(t)+\int_0^t(t-\tau+2)y(\tau)d\tau=t+1$

(1) $y(t)$ (プリントミス)

$y(t)+t\int_0^t y(\tau)\,d\tau+\int_0^t(-\tau+2)y(\tau)d\tau=t+1$

$y'(t)+\int_0^t y(\tau)d\tau+t\,y(t)+(-t+2)y(t)=1$

$y''(t)+y(t)+y(t)+t\,y'(t)-y(t)+(-t+2)y'(t)=0$

$y''(t)+2y'(t)+y(t)=0$

(2) 特性方程式 与式より

$\lambda^2+2\lambda+1=0$ $\lambda=-1$ $y(0)=1$

$\therefore$ $y(t)=(C_1+C_2t)e^{-t}$ $y'(0)=-1$

$\therefore$ $1=C_1$ $C_2-C_1=-1$ $\therefore$ $C_2=0$

$y=e^{-t}$

10.3 $y'=f(x,y)=2xy$

$y_0=1,$ $y_n=1+\int_0^x f(t,y_{n-1}(t))\,dt$ $y_n'=f(x,y_{n-1}(x))$

$y_1=1+\int_0^x 2t\,dt=1+x^2$

$y_2=1+\int_0^x 2t(1+t^2)\,dt=1+x^2+\frac{1}{2}x^4$

$y_3=1+\int_0^x 2t(1+t^2+\frac{1}{2}t^4)\,dt=1+x^2+\frac{1}{2}x^4+\frac{1}{2\cdot3}x^6$

$y_n=1+\frac{1}{2}x^2+\frac{1}{3!}x^6+\quad+\frac{1}{n!}x^{2n}+$

$y=\lim_{n\to\infty}y_n=e^{x^2}$

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§.11 差分方程式

11.1 $f(n)$ $f(0)=5$ $f(1)=11$, $f(n+2)-5f(n+1)+6f(n)=0$

$$f(n+2)-2f(n+1)=3\{f(n+1)-2f(n)\}$$

$$f(n+2)-3f(n+1)=2\{f(n+1)-3f(n)\}$$

$\therefore$ $f(n+1)-2f(n)$ は 公比 3. 初項 1 の等比数列

$f(n+1)-3f(n)$ は 公比 2 初項 -4 の等比数列

$$\therefore\ f(n+2)-2f(n+1)=3^{n+1}$$

$$f(n+2)-3f(n+1)=-4\cdot 2^{n+1}$$

$$\therefore\ f(n+1)=3^{n+1}+4\cdot 2^{n+1}$$

$$\therefore\ f(n)=3^{n}+2^{n+2}$$

§. 12 偏微分方程式

12.1 $0<t<\infty$, $f(t)\in C^\infty$ $\quad z=f(x^2+y^2)$ $\quad z_{xx}+z_{yy}=0$. $f(1)=0$ $\quad f'(1)=-1$

$\frac{d}{dt}f(t)=f'(t)$ $\quad \frac{d^2}{dt^2}f(t)=f''(t)$ とおく $\quad t=x^2+y^2$

$z_x=f'(t)\cdot 2x$ $\qquad z_{xx}=2f'(t)+4x^2f''(t)$

$z_y=f'(t)\cdot 2y$ $\qquad z_{yy}=2f'(t)+4y^2f''(t)$

$z_{xx}+z_{yy}=0$ より $\quad 4(x^2+y^2)f''(t)+4f'(t)=0$

$\therefore\ tf''(t)+f'(t)=0$ $\qquad \frac{f''(t)}{f'(t)}=-\frac{1}{t}$ $\qquad \log|f'(t)|=-\log|t|+C$

$\therefore\ f'(t)=\frac{a}{t}$ $\quad f'(1)=-1$ より $\therefore -1=a$ $\quad \therefore f'(t)=\frac{-1}{t}$

$\therefore\ f(t)=-\log|t|+C$ $\quad f(1)=0$ より $C=0$

$\therefore\ f(t)=-\log t$.

12.2 (1) $f(x,y)\in C^1$ $\quad f_x(x,y)=f_y(x,y)=0$

$(a,b)\in D$ のとき 平均値の定理より

$f(x,y)-f(a,b)=(x-a)f_x(a+\theta(x-a),b+\theta(y-b))+(y-b)f_y(a+\theta(x-a),b+\theta(y-b))$

$=0$

$\therefore f(x,y)=f(a,b)$ 定数

(2) $u(x,y)$, $v(x,y)$ $\quad u_x=v_y$ $\quad v_x=-u_y$, $(u(x,y))^2+v(x,y)^2=$定数

$u^2+v^2=$ 定数より

$2uu_x+2vv_x=0$ $\qquad u_xu-u_yv=0$

$2uu_y+2vv_y=0$ $\qquad u_xv+u_yu=0$

$u^2+v^2\neq 0$ のとき $\quad u_x=u_y=0$ $\quad \therefore v_x=v_y=0$ $\quad \therefore u,v$ は定数

$u^2+v^2=0$ のとき $\quad u=0$ $\quad v=0$ $\quad \therefore u\ v$ は定数

P.77　　§13 総合問題

13.1　$y''+2y'+5y=0$

(1)　$y_1=e^{ax}\sin bx$

$y_1'=e^{ax}(a\sin bx+b\cos bx)$

$y_1''=e^{ax}(a^2\sin bx+2ab\cos bx-b^2\sin bx)$

$y_1''+2y_1'+5y_1$

$=e^{ax}\{(a^2-b^2+2a+5)\sin bx+(2ab+2b)\cos bx\}$

$a^2-b^2+2a+5=0$

$2ab+2b=0$　　$b(a+1)=0$　　$b=0$ のとき a は虚数

$\therefore a=-1 \quad b=\pm 2$

(2)　$y_2=e^{ax}\cos bx$ を代入すると

$y_2'=e^{ax}(a\cos bx-b\sin bx)$

$y_2''=e^{ax}\{(a^2-b^2)\cos bx-2ab\sin bx\}$

$a^2-b^2+2a+5=0$.　$-2ab-2b=0$.

$\therefore a=-1 \quad b=\pm 2$

(3)　$\therefore y_3=C_1y_1+C_2y_2=e^{-x}(C_1\sin 2x+C_2\cos 2x)$

$y_3(0)=1$　　$C_2=1$　　$2C_1-C_2=0$　　$C_1=\frac{1}{2}$

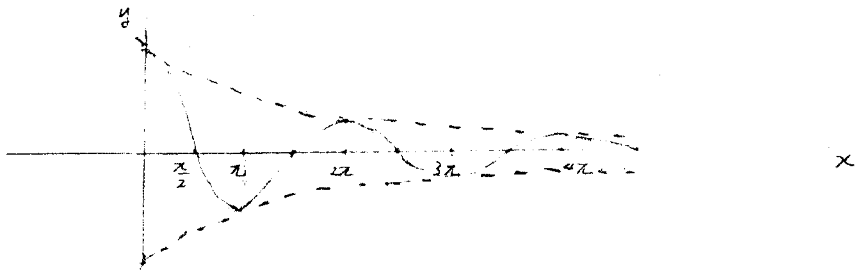
$C_1=\frac{1}{2},\ C_2=1$

(4)　$y_3(x)=e^{-x}(\frac{1}{2}\sin 2x+\cos 2x)$

$y_3'(x)=e^{-x}(-\frac{5}{2}\sin 2x)$

$\therefore y_3'(x)=0$　　$\sin 2x=0$　　$x=\frac{n}{2}\pi$

$y_3(\frac{n\pi}{2})=(-1)^n e^{-\frac{n\pi}{2}}$



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13.2 $\frac{dx}{dt} = -ax$, $\frac{dy}{dt} = ax + by$ $\quad x(0) = 1 \quad y(0) = 0$

$\frac{dx}{x} = -a\,dt \quad \log|x| = -at + c \quad x = Ce^{-at} \quad x(0) = 1$ より

$x = e^{-at}$

$\frac{dy}{dt} = ae^{-at} + by \quad \therefore \frac{dy}{dt} - by = ae^{-at}$

$\therefore y = e^{bt}\left\{ \int ae^{-(a+b)t}dt + c \right\} = -\frac{a}{a+b}e^{-at} + ce^{bt}$

$0 = -\frac{a}{a+b} + c \quad c = \frac{a}{a+b}$

$\therefore y = \frac{a}{a+b}(e^{bt} - e^{-at})$

$\therefore \lim_{t \to \infty} \frac{y}{e^{bt}} = \frac{a}{a+b}$

$\therefore x = e^{-at}, \quad y = \frac{a}{a+b}(e^{bt} - e^{-at}) \quad \lim_{t \to \infty} \frac{y}{e^{bt}} = \frac{a}{a+b}$